



SAGEX

Scattering Amplitudes:
from Geometry to Experiment



Breakthroughs in Quantum Field Theory and Gravity: an Afternoon With Bern, Dixon and Kosower

Queen Mary University of London

7 November 2019

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 764850 (SAGEX)

Gravity as a Double Copy of Gauge Theory and its Applications

November 7, 2019

SAGEX Meeting, QMUL

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SAGEX

Scattering Amplitudes:
from Geometry to Experiment



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Outline

There are many interesting problems in gravity that require perturbative expansions in Newton's constant.

Here I'll describe some ideas for dealing with complexity

1. Complications with gravity perturbation theory.

2. Antidotes:

- Unitarity method.**
- Double copy and color-kinematics duality.**

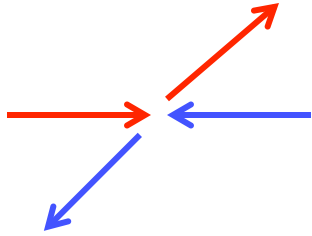
3. Applications:

- Web of theories.**
- Gravitational wave physics.**
- Nonrenormalizability properties of gravity.**

4. Outlook.

Quantum Field Theory and Scattering Amplitudes

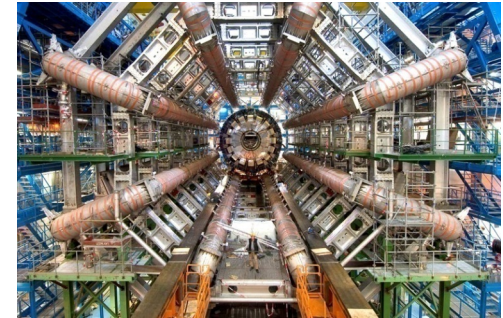
Scattering amplitudes give us quantum mechanical description of events at particle colliders.



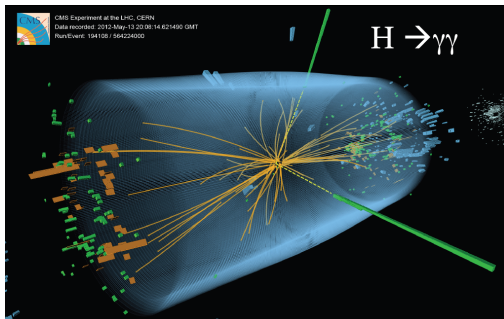
particle scattering



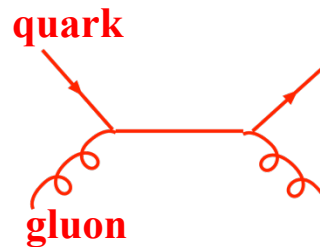
Large Hadron Collider



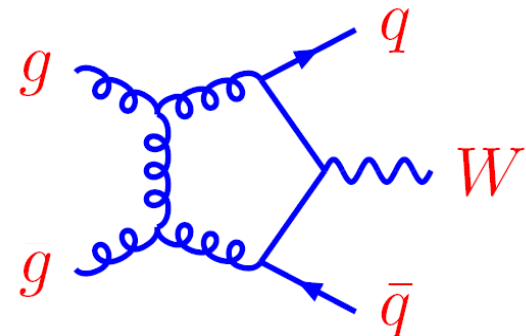
ATLAS Detector



Higgs boson event



Feynman diagram



loop diagram
higher order

Complications with Gravity

Perturbative Gravity


Consider the Einstein gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2} \sqrt{-g} R$$

$\kappa^2 = 32\pi G_{\text{Newton}}$

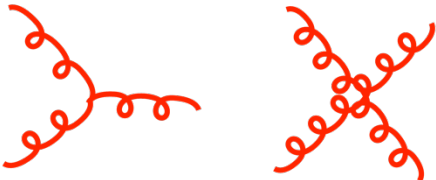
curvature $\rightarrow R$
 Flat-space metric $\rightarrow \eta_{\mu\nu}$
 graviton field $\rightarrow h_{\mu\nu}$
 metric $\rightarrow g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

Infinite number of complicated interactions



Very nonlinear

Compare to gauge-theory Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$


Only three and four point interactions

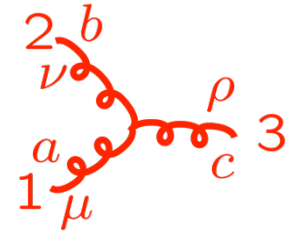
Gravity seems so much more complicated than gauge theory.

Gauge and gravity theories seem rather different.

Three-Point Interactions

Standard perturbative approach:

Three-gluon vertex from strong interactions:



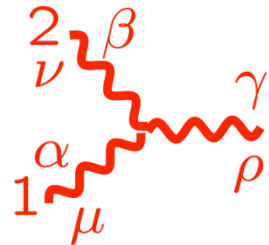
$$V_{3\mu\nu\sigma}^{abc} = -gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_1 - k_2)_\mu + \eta_{\rho\mu}(k_1 - k_2)_\nu)$$

Three-graviton vertex:

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) =$$

$$\begin{aligned} & \text{sym} \left[-\frac{1}{2}P_3(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\sigma\gamma}) + \frac{1}{2}P_3(k_1 \cdot k_2 \eta_{\mu\nu} \eta_{\alpha\beta} \eta_{\sigma\gamma}) \right. \\ & + P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\sigma} \eta_{\beta\gamma}) + 2P_3(k_{1\nu} k_{1\gamma} \eta_{\mu\alpha} \eta_{\beta\sigma}) - P_3(k_{1\beta} k_{2\mu} \eta_{\alpha\nu} \eta_{\sigma\gamma}) \\ & + P_3(k_{1\sigma} k_{2\gamma} \eta_{\mu\nu} \eta_{\alpha\beta}) + P_6(k_{1\sigma} k_{1\gamma} \eta_{\mu\nu} \eta_{\alpha\beta}) + 2P_6(k_{1\nu} k_{2\gamma} \eta_{\beta\mu} \eta_{\alpha\sigma}) \\ & \left. + 2P_3(k_{1\nu} k_{2\mu} \eta_{\beta\sigma} \eta_{\gamma\alpha}) - 2P_3(k_1 \cdot k_2 \eta_{\alpha\nu} \eta_{\beta\sigma} \eta_{\gamma\mu}) \right] \end{aligned}$$





About 100 terms in three vertex


Naïve conclusion: Gravity is a nasty mess.

Feynman Diagrams for Gravity

We will be talking about high order processes

3 loops  $\sim 10^{20}$ TERMS No surprise it has never been calculated via Feynman diagrams.

4 loops  $\sim 10^{26}$ TERMS

5 loops  $\sim 10^{31}$ TERMS More terms than atoms in your brain!

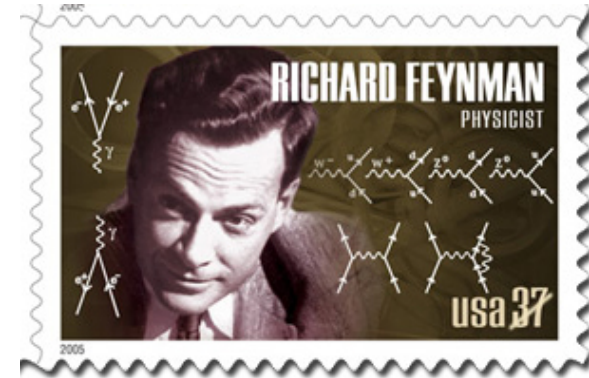
- Calculations to settle this seemed utterly hopeless!
- Seemed destined for dustbin of undecidable questions.

At present there is only one basic approach to carry out such computations, which I will outline.

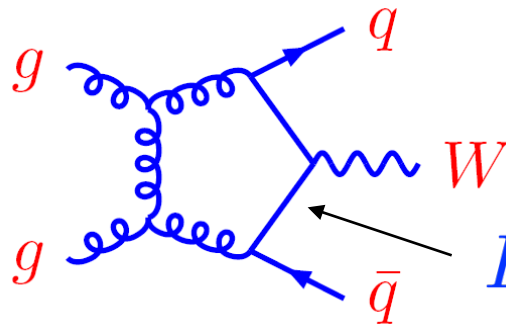
Antidotes to Complexity

Why are Feynman diagrams difficult for high-loop or high-multiplicity processes?

- Vertices and propagators involve unphysical gauge-dependent off-shell states. An important origin of the complexity.



$$\int \frac{d^3\vec{p} dE}{(2\pi)^4}$$



Individual Feynman diagrams unphysical

$$E^2 - \vec{p}^2 \neq m^2$$

Einstein's relation between momentum and energy violated in the loops. **Unphysical states! Not gauge invariant.**

- Use gauge invariant on-shell physical states.
 - On-shell formalism.
 - Don't violate Einstein's relation!
- ZB, Dixon, Dunbar, Kosower (1998)

Scattering Amplitudes Revolution

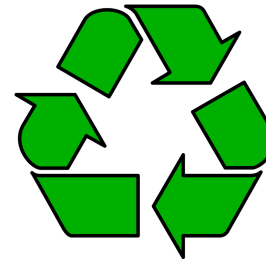
Z. Bern, L. Dixon, D. Kosower,
May 2012 *Scientific American*

Over the years we have developed tools for calculating quantum scattering amplitudes that are “impossibly complicated”

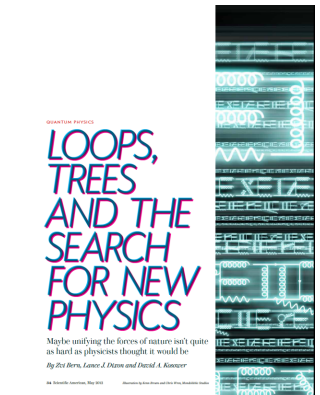
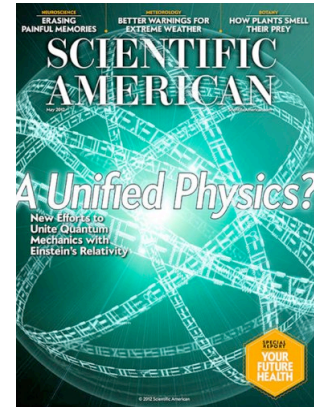
Don't use Lagrangians or Feynman diagrams.

1. Generalized unitarity. Complicated amplitudes assembled from simpler ones. Loops from trees.
2. Double-copy relations. Gravity amplitudes built from much simpler gauge-theory ones.

Key theme: Recycling!



Many more advances, involving also beautiful mathematics most of which I won't discuss here..



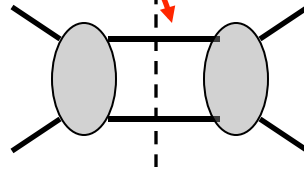
From Tree to Loops: Generalized Unitarity Method

No Feynman rules; no need for virtual particles.

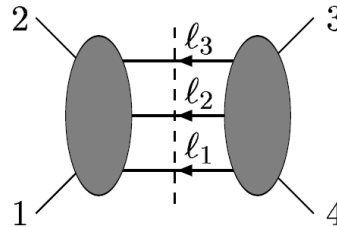
$$E^2 = \vec{p}^2 + m^2 \leftarrow \text{on-shell}$$

ZB, Dixon, Dunbar and Kosower (1994)

Two-particle cut:

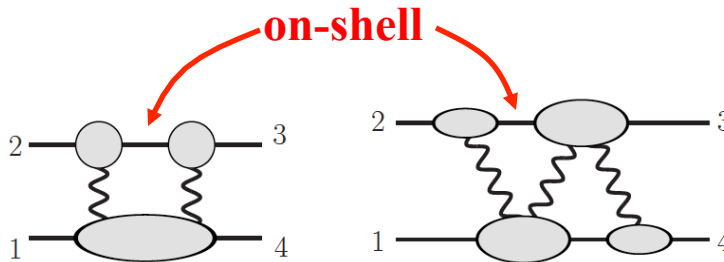


Three-particle cut:



- Systematic assembly of complete loop amplitudes from tree amplitudes.
- Works for any number of particles or loops.

Generalized unitarity as a practical tool for loops.



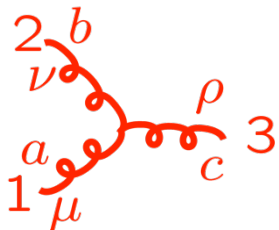
ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala, Pittau, Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger;
ZB, Carrasco, Johansson, Kosower
and many others

**Idea used in the “NLO revolution” in QCD collider physics.
Want to apply it to gravitational wave problem.**

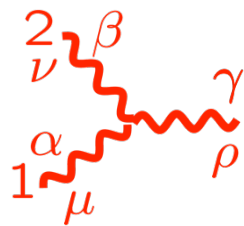
Simplicity of Gravity Amplitudes

People were looking at gravity amplitudes the wrong way.
On-shell viewpoint much more powerful.

On-shell three vertices contains all information: $E_i^2 - \vec{k}_i^2 = 0$

Yang-Mills gauge theory:  $-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$

Einstein gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \\ \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

“square” of
Yang-Mills
vertex.

Very simple interactions.

KLT Relation Between Gravity and Gauge Theory

Kawai-Lewellen-Tye string relations in low-energy limit: KLT (1985)

↙ gravity ↘ gauge-theory color ordered

$$M_4^{\text{tree}}(1, 2, 3, 4) = -is_{12}A_4^{\text{tree}}(1, 2, 3, 4) A_4^{\text{tree}}(1, 2, 4, 3),$$
$$M_5^{\text{tree}}(1, 2, 3, 4, 5) = is_{12}s_{34}A_5^{\text{tree}}(1, 2, 3, 4, 5) A_5^{\text{tree}}(2, 1, 4, 3, 5) \\ + is_{13}s_{24}A_5^{\text{tree}}(1, 3, 2, 4, 5) A_5^{\text{tree}}(3, 1, 4, 2, 5)$$



Generalizes to explicit all-leg form. ZB, Dixon, Perelstein, Rozowsky

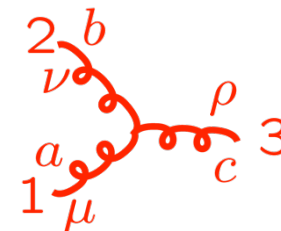
1. Gravity is derivable from gauge theory. Standard Lagrangian methods offers no hint why this is possible.
2. It is very generally applicable.

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (2007)

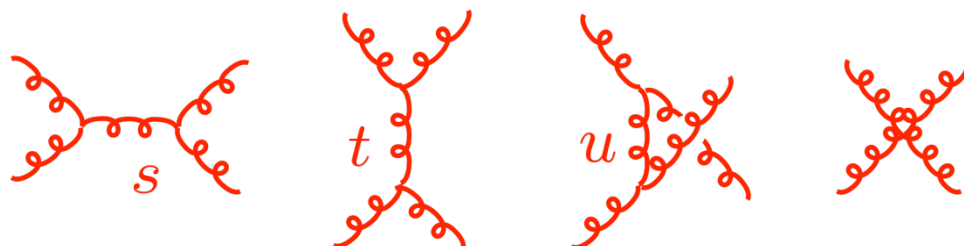
coupling constant \rightarrow color factor \rightarrow momentum dependent kinematic factor

$$-g f^{abc} (\eta_{\mu\nu} (k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use $1 = s/s = t/t = u/u$
to assign 4-point diagram
to others.

$$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2 \\ u = (k_1 + k_3)^2$$

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

Color factors satisfy Jacobi identity:

Numerator factors satisfy similar identity:

$$c_u = c_s - c_t \\ n_u = n_s - n_t$$

Proven at tree level

Duality Between Color and Kinematics

Consider five-point tree amplitude:

ZB, Carrasco, Johansson (BCJ)

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

color factor
kinematic numerator factor
Feynman propagators

$$c_1 = f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} \quad c_2 = f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5} \quad c_3 = f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5 k_2 \cdot \varepsilon_1 \varepsilon_2 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5 + \dots$$

$$c_1 + c_2 + c_3 = 0 \Leftrightarrow n_1 + n_2 + n_3 = 0$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* algebraic constraint equations.

Progress on unraveling relations.

BCJ, Bjerrum-Bohr, Feng, Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer;

Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer

O'Connell and Montiero; Bjerrum-Bohr, Damgaard, O'Connell and Montiero; O'Connell, Montiero, White;

Du, Feng and Teng, Song and Schlotterer, etc.

Higher-Point Gravity and Gauge Theory

ZB, Carrasco, Johansson



**gauge theory
(QCD):**

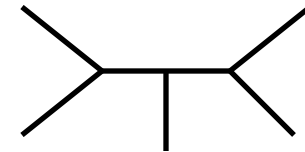
$$\mathcal{A}_n^{\text{tree}} = ig^{n-2} \sum_i \frac{c_i n_i}{D_i}$$

color factor
kinematic numerator factor
Feynman propagators

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

$$c_i \rightarrow n_i$$



sum over diagrams
with only 3 vertices

Einstein gravity:

$$\mathcal{M}_n^{\text{tree}} = i\kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$

$$n_i \sim k_4 \cdot k_5 k_2 \cdot \varepsilon_1 \varepsilon_2 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5 + \dots$$

Gravity and gauge theory kinematic numerators are the same!

Same ideas conjectured to hold at loop level.

Cries out for a unified description of gravity with gauge theory, presumably along the lines of string theory.

Gravity From Gauge Theory

Here we consider only simplest constructions:

$N = 8$ sugra: $(N = 4 \text{ sYM}) \times (N = 4 \text{ sYM})$

$N = 5$ sugra: $(N = 4 \text{ sYM}) \times (N = 1 \text{ sYM})$

$N = 4$ sugra: $(N = 4 \text{ sYM}) \times (N = 0 \text{ sYM})$

Spectrum controlled by simple tensor product of gauge theories.

More sophisticated lower-susy cases: QCD, magical supergravities, Einstein-YM with and without Higgsing, twin supergravities.

Anastasiou, Bornsten, Duff; Duff, Hughs, Nagy; Johansson and Ochirov;
Carrasco, Chiodaroli, Günaydin and Roiban; ZB, Davies, Dennen, Huang and Nohle;
Nohle; Chiodaroli, Günaydin, Johansson, Roiban. A. Anastasiou, L. Borsten, M.J. Duff, M.J. Hughes, Marrani, Nagy, Zoccali.

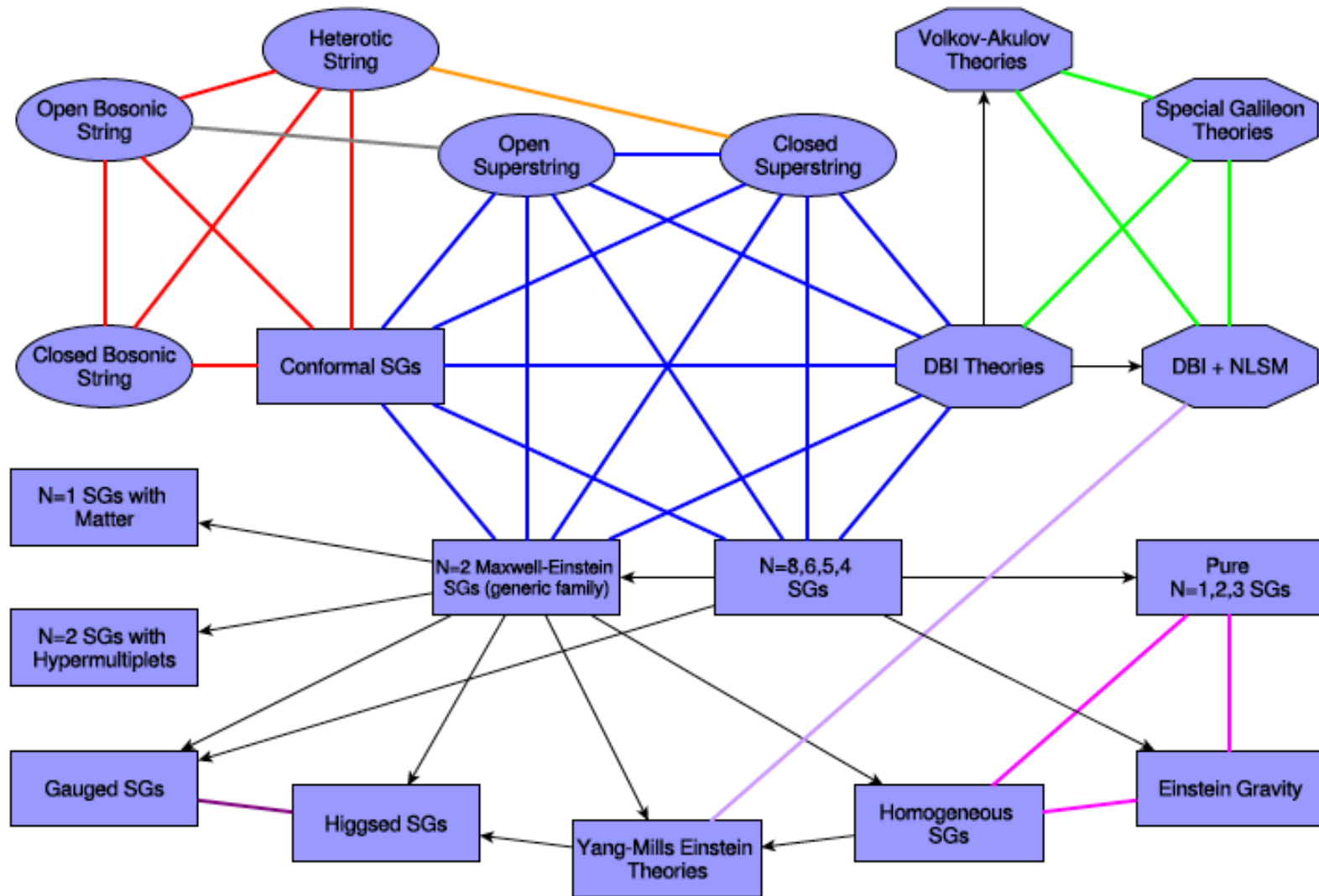
Many other theories in double-copy story, including open and closed string theory, NLSM, Dirac Born Infeld, Galileon and Z theory.

Cachazo, He, Yuan; Chen Du, Broedel, Schlotterer and Stieberger; Carrasco, Mafra, Schlotterer;

Hard to keep track of theories where double copy holds.

Web of Theories

ZB, Carrasco, Chiodaroli, Johansson, Roiban arXiv:1909.01358, Section 5.



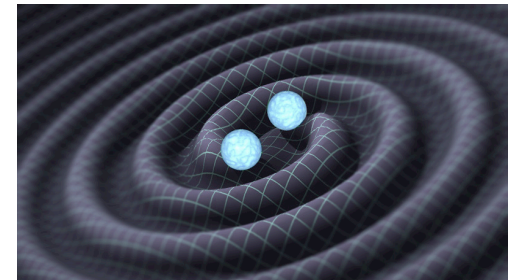
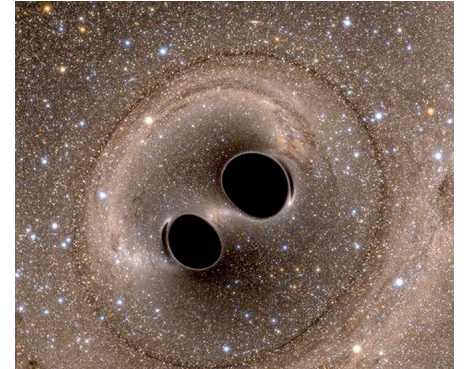
Double copy links various theories through their component theories.

Double Copy for Classical Solutions

Goal is to formulate gravity solutions directly in terms of gauge theory

Variety of special cases:

- Schwarzschild and Kerr black holes.
- Solutions with cosmological constant.
- Radiation from accelerating black hole.
- Maximally symmetric space times.
- Plane wave background.
- Gravitational radiation.



Luna, Monteiro, O'Connell and White;
Luna, Monteiro, NicholSEN, O'Connell and White;
Ridgway and Wise; Carrillo González, Penco, Trodden;
Adamo, Casali, Mason, Nekovar;
Goldberger and Ridgway; Chen;
Luna, Monteiro, Nicholson, Ochirov;
Bjerrum-Bohr, Donoghue, Vanhove;
O'Connell, Westerberg, White; Kosower, Maybee, O'Connell, etc

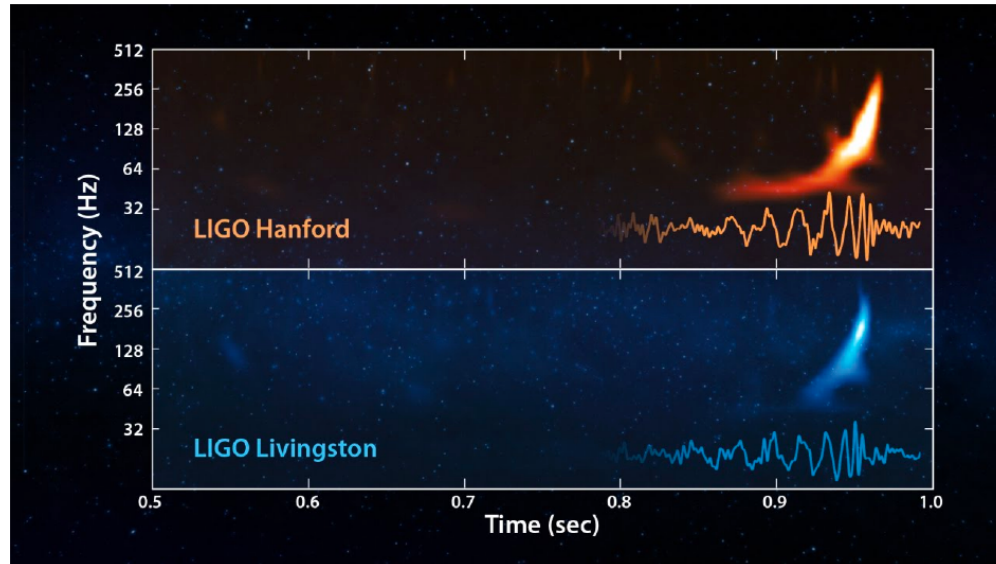
**Still no general understanding.
But plenty of examples.**

Gravitational Waves

See also David Kosower's talk

Outline

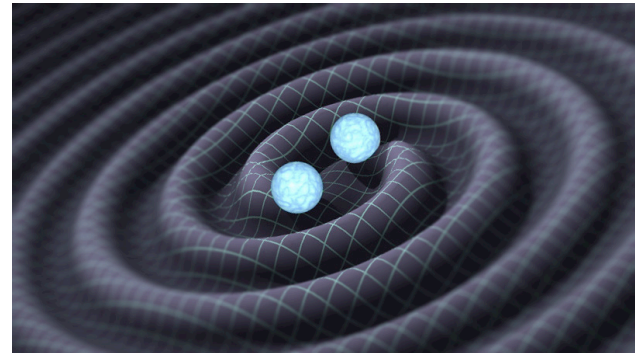
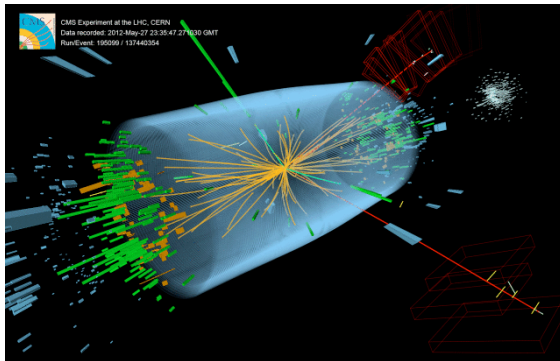
Era of gravitational wave astronomy has begun.



For an instant brighter in gravitational radiation than all the stars in the visible universe are in EM radiation!

Can Particle Theory Help with Gravitational Waves?

What does particle physics have to do with classical dynamics of astrophysical objects?



Black holes and neutron stars are point particles as far as long wavelength radiation is concerned.

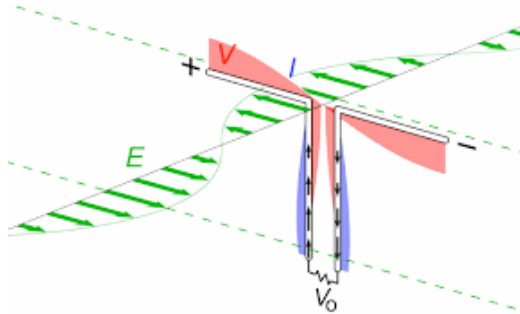
EFT approach: Goldberger, Rothstein, Porto; Vaydia, Foffa , Porto, Rothstein, Sturani; etc

Will explain that the tools that we use in particle physics are ideally suited to push the state of the art in gravitational-wave physics.

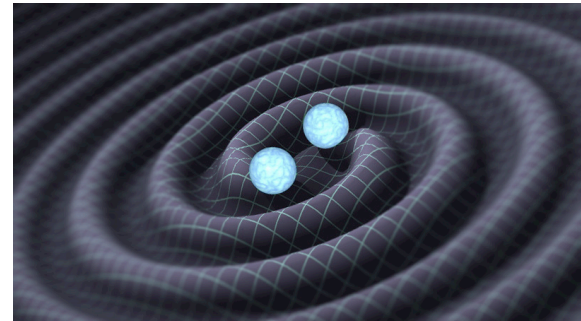
Gravitational Versus Electromagnetic Waves

Gravitational waves might seem mysterious: Einstein and curved space.

Essential idea is no more complicated than for electromagnetic radiation.



dipole antenna



quadrupole antenna

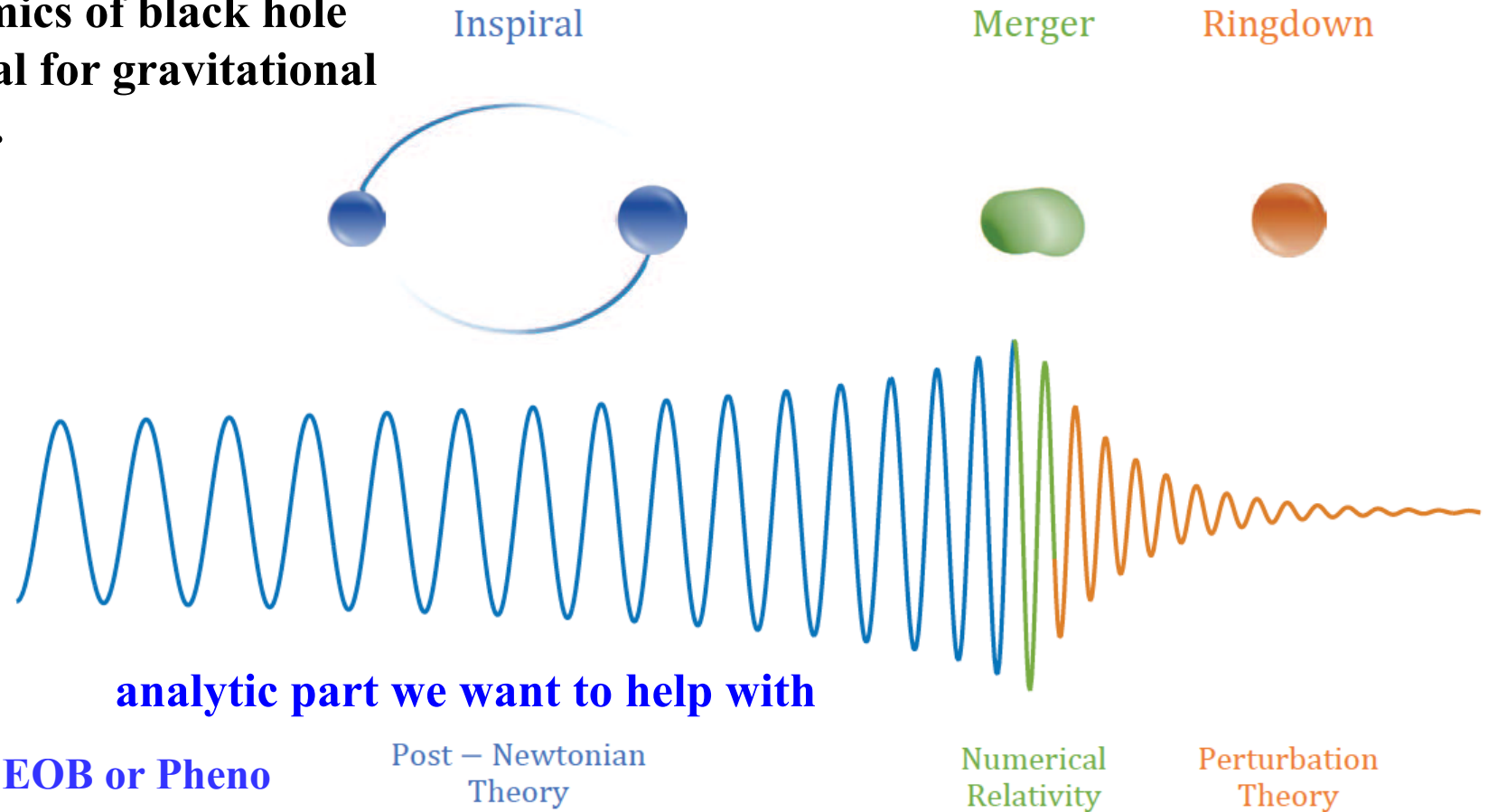
Only real complication is gravity is non-linear. But so is Yang-Mills theory.

Double copy message is clear: If we can compute something in gauge theory we should be able to compute it in gravity.

$$\text{Gravity} \sim (\text{gauge theory})^2$$

Goal: Improve on post-Newtonian Theory

**Dynamics of black hole
inspiral for gravitational
waves.**



Small errors accumulate. Need for high precision.

Post Newtonian Approximation

For orbital mechanics:

Expand in G and v^2

$$v^2 \sim \frac{GM}{R} \ll 1$$



virial theorem

In center of mass frame:

$$m = m_A + m_B, \quad \nu = \mu/M,$$

$$\mu = m_A m_B / m, \quad P_R = P \cdot \hat{R}$$

$$\frac{H}{\mu} = \frac{P^2}{2} - \frac{Gm}{R} \quad \leftarrow \text{Newton}$$

$$+ \frac{1}{c^2} \left\{ -\frac{P^4}{8} + \frac{3\nu P^4}{8} + \frac{Gm}{R} \left(-\frac{P_R^2 \nu}{2} - \frac{3P^2}{2} - \frac{\nu P^2}{2} \right) + \frac{G^2 m^2}{2R^2} \right\}$$

+ ...

1PN: Einstein, Infeld, Hoffmann

Hamiltonian known to 4PN order.

2PN: Ohta, Okamura, Kimura and Hiida.

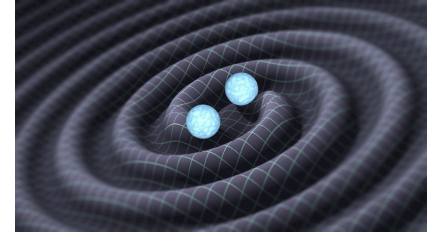
3PN: Damour, Jaranowski and Schaefer; L. Blanchet and G. Faye.

4PN: Damour, Jaranowski and Schaefer; Foffa, Porto, Rothstein, Sturani.

Scattering Amplitudes and Gravitational Radiation

Can we simplify the types of calculations needed for LIGO?

A small industry has developed to study this.



- **Connection to scattering amplitudes.**

Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Luna, Nicholson, O'Connell, White; Guevara; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon; Damour; Bautista, Guevara; Kosower, Maybee, O'Connell; Plefka, Steinhoff, Wormsbecher; Foffa, Mastrolia, Sturani, Sturm; Guevara, Ochirov, Vines; Chung, Huang, Kim, Lee; etc.

- **Worldline approach for radiation and double copy.**

Goldberger and Ridgway; Goldberger, Li, Prabhu, Thompson; Chester; Shen

- **Removing the dilaton contamination.**

Luna, Nicholson, O'Connell, White; Johansson, Ochirov; Johansson, Kalin; Henrik Johansson, Gregor Kälin, Mogull.

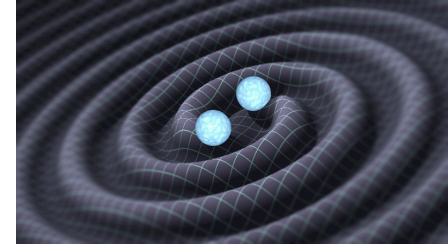
Key Question: Can we calculate something of direct interest to LIGO/Virgo, decisively *beyond* previous state of the art?

Which problem to solve?

ZB, Cheung, Roiban, Shen, Solon, Zeng

Some problems for (analytic) theorists:

1. Spin.
2. Finite size effects.
3. New physics effects.
4. Radiation.



→ 5. High orders in perturbation theory. ←

Which problem should we solve?

- Needs to be extremely difficult using standard methods.
- Needs to be of direct importance to LIGO theorists.
- Needs to be in a form that can in principle enter LIGO analysis pipeline.

2-body Hamiltonian at 3rd post-Minkowskian order

PN versus PM expansion for conservative two-body dynamics

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}}_{\text{non-spinning compact objects}} + \frac{1}{c^2} [\dots] + \frac{1}{c^4} [\dots] + \dots$$

From Buonanno
Amplitudes 2018

$$E(v) = -\frac{\mu}{2} v^2 + \dots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	...
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	...
...						

(credit: Justin Vines)

$$1 \rightarrow Mc^2, \quad v^2 \rightarrow \frac{v^2}{c^2}, \quad \frac{1}{r} \rightarrow \frac{GM}{rc^2}.$$

current known
PN results

current known
PM results

overlap between
PN & PM results

unknown

- PM results (Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep)

Our Approach

ZB, Cheung, Roiban, Shen, Solon, Zeng

Cheung, Rothstein, Solon (2018)

**Amplitudes
community**

**Gravitational
Scattering
Amplitudes**

Kawai, Lewellen, Tye

ZB, Dixon, Dunbar and Kosower

ZB, Dixon, Dunbar, Perelstein, Rozowsky

ZB, Carrasco, Johansson; Etc

**Effective
Field Theory
Methods**

**EFT
community**

Goldberger, Rothstein;

Porto; Neill, Rothstein;

Vaydia, Foffa, Porto, Rothstein, Sturani; etc

**Post
Minkowskian
Potentials**

Inefficient: Start with quantum theory and take $\hbar \rightarrow 0$

Efficient: Almost magical simplifications for gravity amplitudes.
EFT methods efficiently target pieces we want.

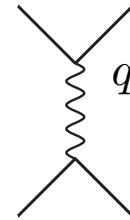
Efficiency wins

2 Body Potentials and Amplitudes

Iwasaki; Gupta, Radford; Donoghue; Holstein, Donoghue; Holstein and A. Ross; Bjerrum-Bohr, Donoghue, Vanhove; Neill, Rothstein; Bjerrum-Bohr, Damgaard, Festuccia, Planté. Vanhove; Chueng, Rothstein, Solon; Chung, Huang, Kim, Lee; etc.

Tree-level: Fourier transform gives classical potential.

$$V(r) = \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} A^{\text{tree}}(\mathbf{q})$$



Beyond 1 loop things quickly become less obvious:



- What you learned in grad school on \hbar and classical limits is wrong! Loops have classical pieces.
- $1/\hbar^L$ scaling of at L loop.
- Double counting and iteration, IR singularities.
- Cross terms between $1/\hbar$ and \hbar .
- Which corners of multiloop integrals are classical? How to extract?

$$e^{iS_{\text{classical}}/\hbar}$$

Piece of loops are classical: Our task is to efficiently extract these pieces.

We harness EFT to clean up confusion

See David Kosower's talk for alternative

EFT is a Clean Approach


**Build EFT from which we can read off potential.
Avoids a variety of confusions related to taking
classical limit.**

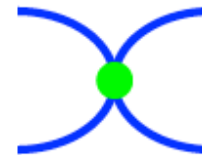
Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

$$L_{\text{kin}} = \int_{\mathbf{k}} A^\dagger(-\mathbf{k}) \left(i\partial_t + \sqrt{\mathbf{k}^2 + m_A^2} \right) A(\mathbf{k}) \\ + \int_{\mathbf{k}} B^\dagger(-\mathbf{k}) \left(i\partial_t + \sqrt{\mathbf{k}^2 + m_B^2} \right) B(\mathbf{k})$$

**A, B scalars
represents spinless
black holes**

$$L_{\text{int}} = - \int_{\mathbf{k}, \mathbf{k}'} V(\mathbf{k}, \mathbf{k}') A^\dagger(\mathbf{k}') A(\mathbf{k}) B^\dagger(-\mathbf{k}') B(-\mathbf{k})$$

 **potential**



$$H = \sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2} + V$$

**2 body Hamiltonian
in c.o.m. frame.**

**Match amplitudes of this theory to the full theory in classical limit to
extract a potential.**

Our friends at LIGO/Virgo want Hamiltonians.

EFT Matching

Cheung, Rothstein, Solon

full general relativity
(complicated)

Amplitude methods
double copy



tree amplitude

generalized
unitarity



loop integrand

loop
integration



GR loop amplitude

effective theory
(simpler)

build
ansatz



potential

Feynman
diagrams



loop integrand

loop
integration



EFT loop amplitude

identical
physics

=

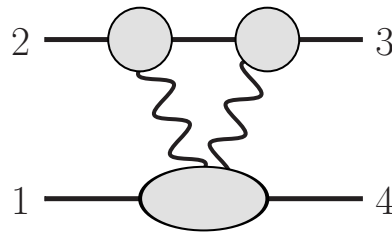
**Roundabout, but efficiently determines potential.
IR singularities cancel trivially.**

Full Theory: Unitarity + Double Copy

- **Long-range force:** Two matter lines must be separated by cut propagators.
- **Classical potential:** 1 matter line per loop is cut (on-shell).

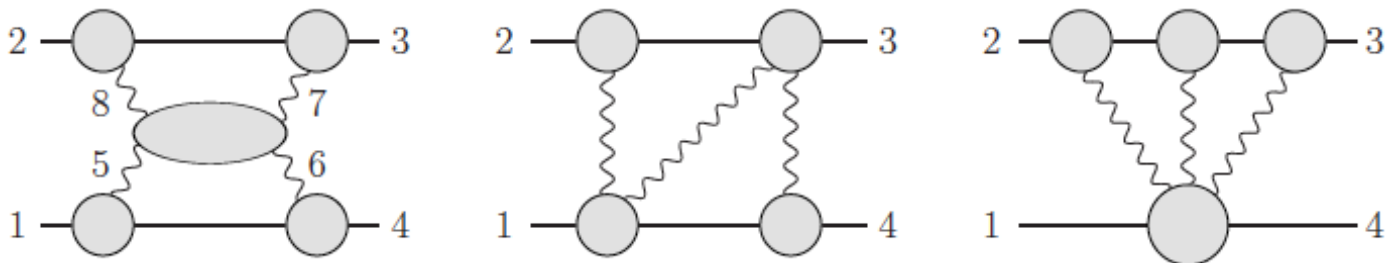
Neill and Rothstein ; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

Only independent unitarity cut for 2 PM.



**exposed lines on-shell (long range).
Classical pieces simple!**

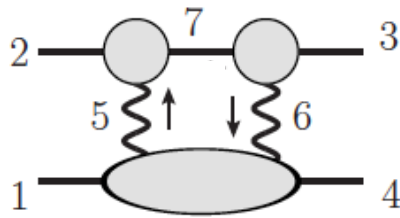
Independent generalized unitarity cuts for 3 PM.



Use our amplitudes tools for this part.

Generalized Unitarity Cuts

Primary means of construction uses BCJ in D dimensions, but KLT with helicity should have better scaling at higher loops and gives compact expressions.



2nd post-Minkowskian order

$$\begin{aligned}
 C_{\text{GR}} &= \sum_{h_5, h_6 = \pm} M_3^{\text{tree}}(3^s, 6^{h_6}, -7^s) M_3^{\text{tree}}(7^s, -5^{h_5}, 2^s) M_4^{\text{tree}}(1^s, 5^{-h_5}, -6^{-h_6}, 4^s) \\
 &= \sum_{h_5, h_6 = \pm} i t [A_3^{\text{tree}}(3^s, 6^{h_6}, -7^s) A_3^{\text{tree}}(7^s, -5^{h_5}, 2^s) A_4^{\text{tree}}(1^s, 5^{-h_5}, -6^{-h_6}, 4^s)] \\
 &\quad \times [A_3^{\text{tree}}(3^s, 6^{h_6}, -7^s) A_3^{\text{tree}}(7^s, -5^{h_5}, 2^s) A_4^{\text{tree}}(4^s, 5^{-h_5}, -6^{-h_6}, 1^s)]
 \end{aligned}$$

Problem of computing the generalized cuts in gravity is reduced to multiplying and summing gauge-theory tree amplitudes.

Gauge-Theory Warm-up

$$A^{\text{tree}}(1^s, 2^+, 3^+, 4^s) = i \frac{m_1^2 [23]}{\langle 23 \rangle t_{12}}$$

$$A^{\text{tree}}(1^s, 2^+, 3^-, 4^s) = -i \frac{\langle 3|1|2 \rangle^2}{\langle 23 \rangle [23] t_{12}}$$

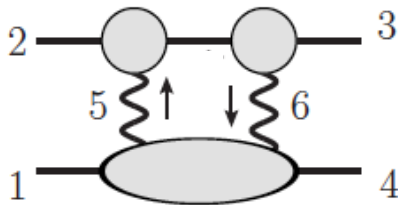


color-ordered gauge-theory
tree amplitudes

- This is all you need for 2 PM.
- Scaling with number of loops is very good.

$$s_{23} = (p_2 + p_3)^2$$

$$t_{ij} = 2p_i \cdot p_j$$



$$C_{\text{YM}} = 2 \left(\frac{\mathcal{E}^2 + \mathcal{O}^2}{s_{23}^2} + m_1^2 m_2^2 \right) \frac{1}{t_{15}}$$

sum over helicities
gauge theory

$$\mathcal{E}^2 = \frac{1}{4} s_{23}^2 (t_{15} - t_{12})^2$$

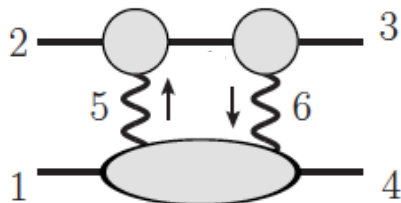
$$\mathcal{O}^2 = \mathcal{E}^2 - s_{23} m_2^2 (s_{23} m_1^2 + s_{23} t_{15} + t_{15}^2)$$

Simple cut contains all information we need for 2PM Hamiltonian

One loop gravity warmup

$$s_{23} = (p_2 + p_3)^2$$

$$t_{ij} = 2p_i \cdot p_j$$



**Apply unitarity and KLT relations.
Import gauge-theory results.**

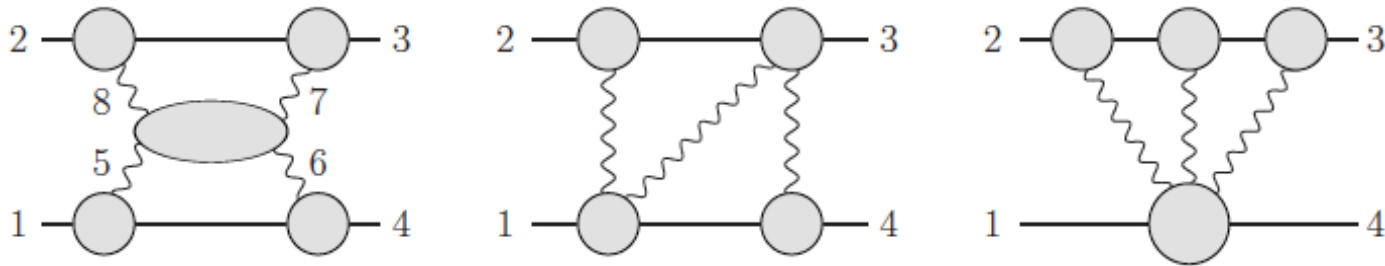
$$C_{\text{GR}} = 2 \left[\frac{1}{t^4} (\mathcal{E}^4 + \mathcal{O}^4 + 6\mathcal{E}^2 \mathcal{O}^2) + m_1^4 m_2^4 \right] \left[\frac{1}{t_{15}} + \frac{1}{t_{45}} \right]$$

- Same building blocks as gauge theory.
- Double copy is visible even though we removed dilaton and axion.

We can extract classical scattering angles or potentials following 2 PM literature.

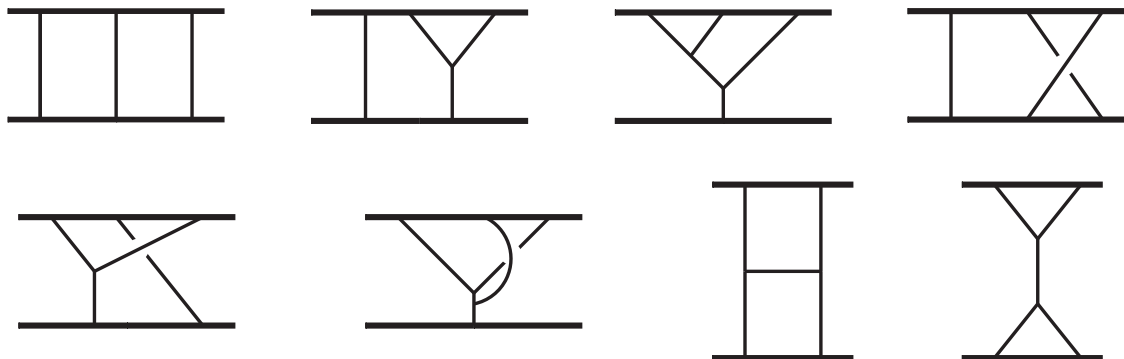
Damour; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove;
Cheung, Rothstein, Solon; Koemans Collado, Di Vecchia, Russo;
etc.

Two Loops and 3 PM



- Somewhat more complicated than previous one loop, but no problem.
- Evaluated using both BCJ and KLT double copies.
- To interface easily with EFT approach, merged unitarity cuts into Feynman-like diagrams to get integrand.

Integrand organized into 8 independent diagrams that may contribute in classical limit:



Integration + Extraction of Potential

To integrate, follow path of Cheung, Rothstein and Solon.

- Efficiently targets the classical pieces of integrals we want.
- Integrals reduce via energy residues to 3 dimensional integrals.

$$\ell_0 \ll |\vec{\ell}| \ll |\vec{p}| \ll m_i$$

potential classical nonrelativistic

Integrals much simpler than for full quantum theory

Detail are found in our paper.

Amplitude in Classical Potential Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)

Classical limit. The $O(G^3)$ or 3PM terms are:

rapidity 

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log q^2}{6\gamma^2 \xi} \left[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu(3 + 12\sigma^2 - 4\sigma^4) \operatorname{arcsinh} \sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}} \right. \\ \left. - \frac{18\nu\gamma(1 - 2\sigma^2)(1 - 5\sigma^2)}{(1 + \gamma)(1 + \sigma)} \right] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi} \left[3\gamma(1 - 2\sigma^2)(1 - 5\sigma^2)F_1 - 32m^2\nu^2(1 - 2\sigma^2)^3 F_2 \right]$$

$$m = m_A + m_B, \quad \mu = m_A m_B / m, \quad \nu = \mu / m, \quad \gamma = E / m, \\ \xi = E_1 E_2 / E^2, \quad E = E_1 + E_2, \quad \sigma = p_1 \cdot p_2 / m_1 m_2,$$

F_1 and F_2 IR divergent iteration terms that don't affect potential.

Amplitude containing classical potential is remarkably simple!

Conservative 3PM Hamiltonian

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)

The 3PM Hamiltonian:

$$H(\mathbf{p}, \mathbf{r}) = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + V(\mathbf{p}, \mathbf{r})$$

$$V(\mathbf{p}, \mathbf{r}) = \sum_{i=1}^3 c_i(\mathbf{p}^2) \left(\frac{G}{|\mathbf{r}|} \right)^i,$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi} (1 - 2\sigma^2), \quad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi} \left[\frac{3}{4} (1 - 5\sigma^2) - \frac{4\nu\sigma (1 - 2\sigma^2)}{\gamma\xi} - \frac{\nu^2 (1 - \xi) (1 - 2\sigma^2)^2}{2\gamma^3 \xi^2} \right],$$

$$c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi} \left[\frac{1}{12} (3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3) - \frac{4\nu (3 + 12\sigma^2 - 4\sigma^4) \operatorname{arcsinh} \sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}} \right. \\ \left. - \frac{3\nu\gamma (1 - 2\sigma^2) (1 - 5\sigma^2)}{2(1 + \gamma)(1 + \sigma)} - \frac{3\nu\sigma (7 - 20\sigma^2)}{2\gamma\xi} - \frac{\nu^2 (3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2) (1 - 2\sigma^2)}{4\gamma^3 \xi^2} \right. \\ \left. + \frac{2\nu^3 (3 - 4\xi)\sigma (1 - 2\sigma^2)^2}{\gamma^4 \xi^3} + \frac{\nu^4 (1 - 2\xi) (1 - 2\sigma^2)^3}{2\gamma^6 \xi^4} \right],$$

$$m = m_A + m_B, \quad \mu = m_A m_B / m, \quad \nu = \mu / m, \quad \gamma = E / m,$$

$$\xi = E_1 E_2 / E^2, \quad E = E_1 + E_2, \quad \sigma = \mathbf{p}_1 \cdot \mathbf{p}_2 / m_1 m_2,$$

Checks

Primary check:

ZB, Cheung, Roiban, Shen, Solon, Zeng

Compare to 4PN Hamiltonian of Damour, Jaranowski, Schäfer, used in high precision template construction.

Need canonical transformation:

$$(\mathbf{r}, \mathbf{p}) \rightarrow (\mathbf{R}, \mathbf{P}) = (A \mathbf{r} + B \mathbf{p}, C \mathbf{p} + D \mathbf{r})$$

$$A = 1 - \frac{Gm\nu}{2|\mathbf{r}|} + \dots, \quad B = \frac{G(1 - 2/\nu)}{4m|\mathbf{r}|} \mathbf{p} \cdot \mathbf{r} + \dots$$

$$C = 1 + \frac{Gm\nu}{2|\mathbf{r}|} + \dots, \quad D = -\frac{Gm\nu}{2|\mathbf{r}|^3} \mathbf{p} \cdot \mathbf{r} + \dots,$$

Our Hamiltonian equivalent to 4PN Hamiltonian on overlap.

Additional (somewhat redundant) tests:

1. Classical scattering angle matches 4PN result in overlap Bini and Damour

2. Amplitude using 4PN potentials matches result in overlap.
Damour, Jaranowski, Schaefer

3. In test mass limit, $m_1 \ll m_2$, matches Schwarzschild Hamiltonian.
Wex and Schaefer

4 PN Hamiltonian

Damour, Jaranowski, Schaefer


$$\mathbf{n} = \hat{\mathbf{r}}$$

$$\hat{H}_N(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r},$$

$$c^2 \hat{H}_{1PN}(\mathbf{r}, \mathbf{p}) = \frac{1}{8}(3\nu - 1)(\mathbf{p}^2)^2 - \frac{1}{2} \left\{ (3 + \nu)\mathbf{p}^2 + \nu(\mathbf{n} \cdot \mathbf{p})^2 \right\} \frac{1}{r} + \frac{1}{2r^2},$$

$$c^4 \hat{H}_{2PN}(\mathbf{r}, \mathbf{p}) = \frac{1}{16} (1 - 5\nu + 5\nu^2) (\mathbf{p}^2)^3 + \frac{1}{8} \left\{ (5 - 20\nu - 3\nu^2) (\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n} \cdot \mathbf{p})^2 \mathbf{p}^2 - 3\nu^2(\mathbf{n} \cdot \mathbf{p})^4 \right\} \frac{1}{r} \\ + \frac{1}{2} \left\{ (5 + 8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n} \cdot \mathbf{p})^2 \right\} \frac{1}{r^2} - \frac{1}{4}(1 + 3\nu) \frac{1}{r^3},$$

$$c^6 \hat{H}_{3PN}(\mathbf{r}, \mathbf{p}) = \frac{1}{128} (-5 + 35\nu - 70\nu^2 + 35\nu^3) (\mathbf{p}^2)^4 + \frac{1}{16} \left\{ (-7 + 42\nu - 53\nu^2 - 5\nu^3) (\mathbf{p}^2)^3 \right. \\ \left. + (2 - 3\nu)\nu^2(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^2 + 3(1 - \nu)\nu^2(\mathbf{n} \cdot \mathbf{p})^4 \mathbf{p}^2 - 5\nu^3(\mathbf{n} \cdot \mathbf{p})^6 \right\} \frac{1}{r} \\ + \left\{ \frac{1}{16} (-27 + 136\nu + 109\nu^2) (\mathbf{p}^2)^2 + \frac{1}{16}(17 + 30\nu)\nu(\mathbf{n} \cdot \mathbf{p})^2 \mathbf{p}^2 + \frac{1}{12}(5 + 43\nu)\nu(\mathbf{n} \cdot \mathbf{p})^4 \right\} \frac{1}{r^2} \\ + \left\{ \left(-\frac{25}{8} + \left(\frac{\pi^2}{64} - \frac{335}{48} \right) \nu - \frac{23\nu^2}{8} \right) \mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4} \right) \nu(\mathbf{n} \cdot \mathbf{p})^2 \right\} \frac{1}{r^3} + \left\{ \frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2 \right) \nu \right\} \frac{1}{r^4},$$

G^4 

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$\mathbf{n} = \hat{\mathbf{r}}$

$$\begin{aligned}
 c^8 \hat{H}_{4\text{PN}}^{\text{local}}(\mathbf{r}, \mathbf{p}) = & \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4 \right) (\mathbf{p}^2)^5 \\
 & + \left\{ \frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4 \nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n} \cdot \mathbf{p})^4(\mathbf{p}^2)^2 \right) \nu^2 \right. \\
 & + \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n} \cdot \mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n} \cdot \mathbf{p})^6 \mathbf{p}^2 + \frac{35}{256}(\mathbf{n} \cdot \mathbf{p})^8 \right) \nu^3 \\
 & + \left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n} \cdot \mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n} \cdot \mathbf{p})^6 \mathbf{p}^2 - \frac{35}{128}(\mathbf{n} \cdot \mathbf{p})^8 \right) \nu^4 \left. \right\} \frac{1}{r} \\
 & + \left\{ \frac{13}{8}(\mathbf{p}^2)^3 + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n} \cdot \mathbf{p})^4 \mathbf{p}^2 + \frac{369}{160}(\mathbf{n} \cdot \mathbf{p})^6 \right) \nu \right. \\
 & + \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^2 + \frac{9475}{768}(\mathbf{n} \cdot \mathbf{p})^4 \mathbf{p}^2 - \frac{1151}{128}(\mathbf{n} \cdot \mathbf{p})^6 \right) \nu^2 \\
 & + \left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n} \cdot \mathbf{p})^4 \mathbf{p}^2 + \frac{10353}{1280}(\mathbf{n} \cdot \mathbf{p})^6 \right) \nu^3 \left. \right\} \frac{1}{r^2} \\
 & + \left\{ \frac{105}{32}(\mathbf{p}^2)^2 + \left(\left(\frac{2749\pi^2}{8192} - \frac{589189}{19200} \right) (\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024} \right) (\mathbf{n} \cdot \mathbf{p})^2 \mathbf{p}^2 + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280} \right) (\mathbf{n} \cdot \mathbf{p})^4 \right) \nu \right. \\
 & + \left(\left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800} \right) (\mathbf{p}^2)^2 + \left(-\frac{127}{3} - \frac{4035\pi^2}{2048} \right) (\mathbf{n} \cdot \mathbf{p})^2 \mathbf{p}^2 + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384} \right) (\mathbf{n} \cdot \mathbf{p})^4 \right) \nu^2 \\
 & + \left(-\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n} \cdot \mathbf{p})^2 \mathbf{p}^2 - \frac{381}{128}(\mathbf{n} \cdot \mathbf{p})^4 \right) \nu^3 \left. \right\} \frac{1}{r^3} \\
 & + \left\{ \frac{105}{32} \mathbf{p}^2 + \left(\left(\frac{185761}{19200} - \frac{21837\pi^2}{8192} \right) \mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576} \right) (\mathbf{n} \cdot \mathbf{p})^2 \right) \nu \right. \\
 & + \left(\left(\frac{672811}{19200} - \frac{158177\pi^2}{49152} \right) \mathbf{p}^2 + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840} \right) (\mathbf{n} \cdot \mathbf{p})^2 \right) \nu^2 \left. \right\} \frac{1}{r^4} \longleftarrow G^4 \\
 & + \left\{ -\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400} \right) \nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45} \right) \nu^2 \right\} \frac{1}{r^5} \longleftarrow G^5
 \end{aligned}$$

After canonical transformation we match all but G^4 and G^5 terms

Mess is partly due to their gauge choice.

Ours is all orders in p at G^3

Tests of Our 3PM Hamiltonian for LIGO

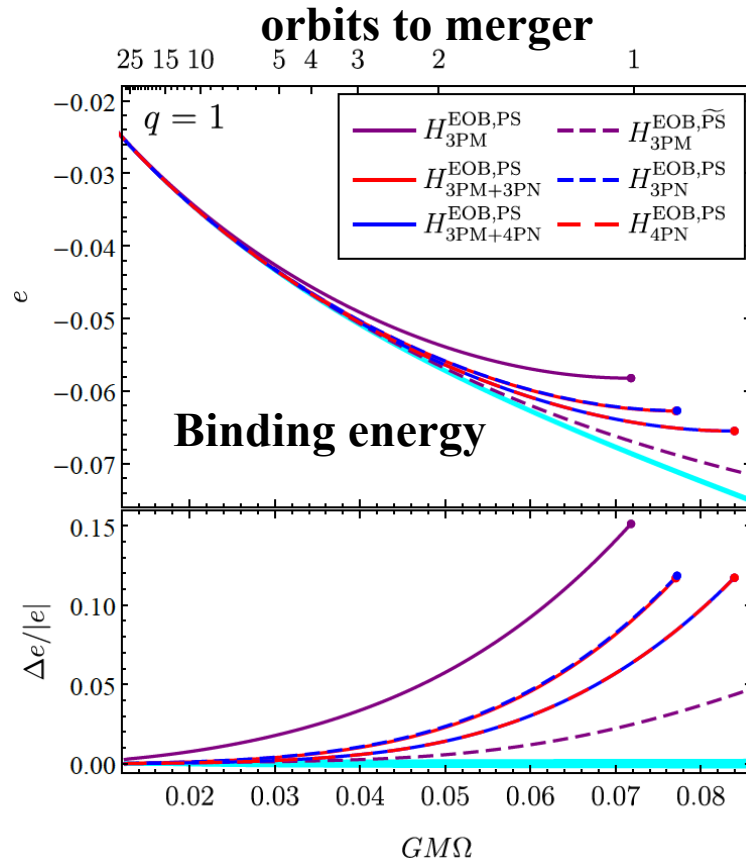
Antonelli, Buonanno, Steinhoff, van de Meent, and Vines, arXiv:1901.07102

(8 days after our paper!)

**Fed into EOB models, which
are needed for good agreement.**

Test against numerical relativity.

**Note: Not conclusive, e. g.
radiation not taken into account**



**Winning curve is based on feeding
our 3PM through one of the EOB models.
Main message: LIGO theorists interested.**

numerical relativity taken as truth

“This rather encouraging result motivates a more comprehensive study...”

Our GR friends are definitely interested! This is only the beginning.

Outlook

- Most exciting part is that methods are far from exhausted.
- Methods scale well to higher orders.
- Started working on 4th PM order. Methods certainly look up to the task.
- Greater improvements on horizon.

Obvious topics to investigate:

- Higher orders. Resummation in G .
- Radiation.
- Spin.
- Finite size effects.



Expect many more advances in coming years.

Understanding UV of Gravity

Quantum Gravity

Often repeated statement:

“Einstein’s theory of General Relativity is incompatible with quantum mechanics.”

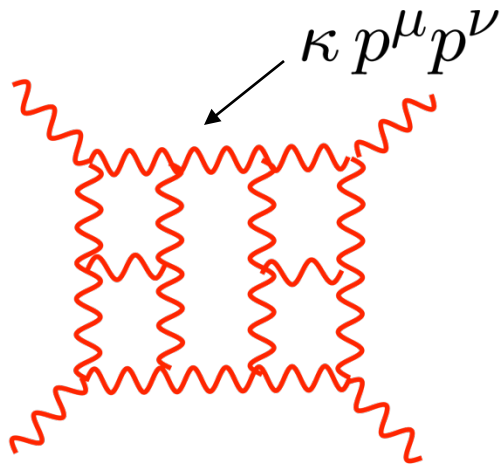
To a large extent this is based on another often repeated statement:

“All point-like quantum theories of gravity are divergent and non-renormalizable.”

Where do these statements come from and are they true?

UV Behavior of Gravity?

$$\kappa = \sqrt{32\pi G_N} \quad \leftarrow \text{Dimensionful coupling}$$



Gravity:
$$\int \prod_{i=1}^L \frac{d^D p_i}{(2\pi)^D} \frac{\cdots \kappa p_j^\mu p_j^\nu \cdots}{\text{propagators}}$$

Gauge theory:
$$\int \prod_{i=1}^L \frac{d^D p_i}{(2\pi)^D} \frac{\cdots g p_j^\nu \cdots}{\text{propagators}}$$

- Extra powers of loop momenta in numerator means integrals are badly behaved in the UV and must diverge at some loop order.
- Much more sophisticated power counting in supersymmetric theories but this is basic idea.

- With more supersymmetry expect better UV properties.
- Need to worry about “hidden cancellations”.
- $N = 8$ supergravity *best* theory to study.

$N = 8$ supergravity: Where is First $D = 4$ UV Divergence?

3 loops $N = 8$	Green, Schwarz, Brink (1982); Howe and Stelle (1989); Marcus and Sagnotti (1985)	X
5 loops $N = 8$	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998); Howe and Stelle (2003,2009)	X
6 loops $N = 8$	Howe and Stelle (2003)	X
7 loops $N = 8$	Grisaru and Siegel (1982); Bossard, Howe, Stelle (2009); Vanhove; Björnsson, Green (2010); Kiermaier, Elvang, Freedman (2010); Ramond, Kallosh (2010); Biesert et al (2010); Bossard, Howe, Stelle, Vanhove (2011)	?
3 loops $N = 4$	Bossard, Howe, Stelle, Vanhove (2011)	X
4 loops $N = 5$	Bossard, Howe, Stelle, Vanhove (2011)	X
4 loops $N = 4$	Vanhove and Tourkine (2012)	✓
9 loops $N = 8$	Berkovits, Green, Russo, Vanhove (2009)	X

“shut up and calculate”

ZB, Kosower, Carrasco, Dixon, Johansson, Roiban; ZB, Davies, Dennen, A. Smirnov, V. Smirnov; series of calculations.

This is what we are most interested in.

Weird structure. Anomaly-like behavior of divergence.

Retracted, but perhaps to be unretracted.

- Track record of predictions from standard symmetries not great.
- Conventional wisdom holds that it will diverge sooner or later.

Supersymmetry and Ultraviolet Divergences

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Björnsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; Bossard, Howe, Stelle, Vanhove, etc

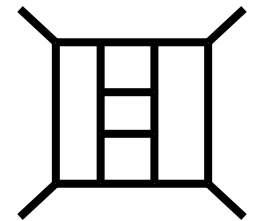
- **First quantized formulation of Berkovits' pure-spinor formalism.** Björnsson and Green
- **Key point: *all* supersymmetry cancellations are exposed.**

Poor UV behavior, unless new types of cancellations between diagrams exist that are “not consequences of supersymmetry in any conventional sense”

“Since we have not evaluated the precise values of the coefficients the possibility of terms vanishing or cancellations between different contributions to the amplitude cannot be ruled out.”

Björnsson and Green

- $N = 8$ sugra should diverge at 5 loops in $D = 24/5$.
- $N = 8$ sugra should diverge at 7 loops in $D = 4$.

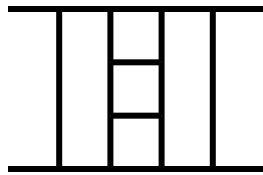


Consensus agreement from all power-counting methods, including our unitarity approaches.

Finding BCJ Forms Nontrivial

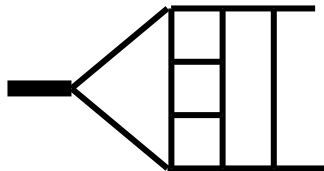
Gravity integrands might be “free”, but gauge-theory ones are not.
Trouble beyond four loops.

5-loop 4-pt $N = 4$ sYM amplitude:



Despite considerable effort no one has succeeded in finding a BCJ form.

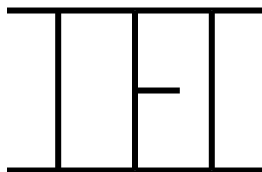
$N = 4$ sYM 5 loop form factor:



On other hand, no trouble with form factor.

Gang Yang (2016)

Two-loop five-point QCD identical helicity:



This required an ansatz with curiously high power counting.

O’Connell and Mogull (2015)

It can be difficult to find BCJ representations.

Generalized Double Copy

ZB, Carrasco, Chen, Johansson, Roiban, Zeng (2017)

Task is to convert $N = 4$ sYM 5-loop integrand into $N = 8$ sugra.

BCJ representation hard to find and KLT on cuts too complicated.

Generalized Double Copy:

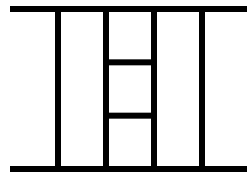
Start with “naïve double copy” of *any* correct sYM integrand:

$C_i \rightarrow n_i$ *Not a BCJ representation*

Without BCJ duality, *not* the correct $N = 8$ integrand

$N = 8$ cuts:

Max cuts:

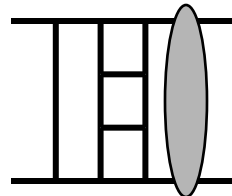


Automatic

Generalized Unitarity

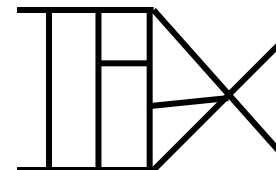
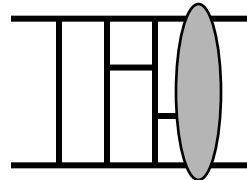
**All exposed legs
on shell**

N_{max} cuts:



Automatic via BCJ, 4pt trees always work

N^2_{max} cuts:



**Add contact term
to make it work**

Double copy construction for any gauge-theory representation!

$N = 8$ UV at Five loops

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

In $D = 24/5$ we obtain a divergence:

Basis of UV divergent
integrals

$$\mathcal{M}_4^{(5)} \Big|_{\text{leading}} = -\frac{16 \times 629}{25} \left(\frac{\kappa}{2}\right)^{12} (s^2 + t^2 + u^2)^2 stu M_4^{\text{tree}} \left(\frac{1}{48} \text{Box} + \frac{1}{16} \text{Cross} \right)$$

With hindsight it is *easy* to foresee this result, except if it turned out to be zero.

“Impossible” gravity calculations are pretty standard by now and shed nontrivial new light on UV properties.

Scorecard on Symmetry Predictions

- $N = 4$ sugra should diverge at 3 loops in $D = 4$. ✗
- $N = 5$ sugra should diverge at 4 loops in $D = 4$. ✗
- Half maximal sugra diverges at 2 loops in $D = 5$. ✗
- $N = 8$ sugra should diverge at 5 loops in $D = 24/5$. ✓
- $N = 8$ sugra should diverge at 7 loops in $D = 4$. ? **key question**

ZB, Davies, Dennen (2012, 2014); ZB, Davies, Dennen, Huang(2012)

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

- **UV cancellation of $N = 5$ supergravity at 4 loops in $D = 4$ definite mystery. Clear problem with standard symmetry arguments.**

Freedman, Kallosh and Yamada (2018)

What is the difference between $N = 5$ and $N = 8$?

$D = 4$ has extra cancellations.

Edison, Herrmann, Parra-Martinez, Trnka (2019)

Goal is to provide definitive answers. Need to go to 7 loops!

Higher-loop Structure.

Green Schwarz, Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Carrasco, Dixon, Johansson, Roiban; ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng

Over the years we've obtained results for $N = 8$ sugra through five loops. Vacuum diagrams capture UV divergences.

dots represent extra propagators

$$\mathcal{M}_4^{(1)} \Big|_{\text{leading}} = -3 \mathcal{K}_G \left(\frac{\kappa}{2} \right)^4 \text{ (circle with 4 dots) }, \quad D_c = 8$$

$$\mathcal{M}_4^{(2)} \Big|_{\text{leading}} = -8 \mathcal{K}_G \left(\frac{\kappa}{2} \right)^6 (s^2 + t^2 + u^2) \left(\frac{1}{4} \text{ (circle with vertical line and 4 dots) } + \frac{1}{4} \text{ (circle with horizontal line and 4 dots) } \right), \quad D_c = 7$$

$$\mathcal{M}_4^{(3)} \Big|_{\text{leading}} = -60 \mathcal{K}_G \left(\frac{\kappa}{2} \right)^8 stu \left(\frac{1}{6} \text{ (circle with 3 lines from center to boundary and 3 dots) } + \frac{1}{2} \text{ (circle with 3 lines from center to boundary and 3 dots) } \right), \quad D_c = 6$$

$$\mathcal{M}_4^{(4)} \Big|_{\text{leading}} = -\frac{23}{2} \mathcal{K}_G \left(\frac{\kappa}{2} \right)^{10} (s^2 + t^2 + u^2)^2 \left(\frac{1}{4} \text{ (circle with triangle and 3 dots) } + \frac{1}{2} \text{ (circle with triangle and 3 dots) } + \frac{1}{4} \text{ (circle with triangle and 3 dots) } \right), \quad D_c = \frac{11}{2}$$

$$\mathcal{M}_4^{(5)} \Big|_{\text{leading}} = -\frac{16 \times 629}{25} \mathcal{K}_G \left(\frac{\kappa}{2} \right)^{12} (s^2 + t^2 + u^2)^2 \left(\frac{1}{48} \text{ (circle with square and 4 dots) } + \frac{1}{16} \text{ (circle with X and 4 dots) } \right), \quad D_c = \frac{24}{5}$$

We now have a lot of theoretical “data” to guide us to 6,7 loops.

Summary

- Particle physics give us new ways to think about problems of current interest in general relativity.
- Double-copy idea gives a unified framework for gravity and gauge theory.
- Sample applications:
 - Web of theories.
 - High loop order explorations of supergravity, especially UV.
 - 2 body Hamiltonians for gravitational radiation.

Expect many more advances in coming years, not only for gravitational-wave physics, but more generally for understanding gravity and its relation to the other forces via double copy.

Further Reading

Double Copy:

Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson, R. Roiban.
“The Duality Between Color and Kinematics and its Applications”
arXiv:1909.01358

Black Hole Physics from Amplitudes:

Z. Bern, C. Cheung, R. Roiban, Chia-Hsien Shen, M. P. Solon, M. Zeng
“Black Hole Binary Dynamics from the Double Copy and Effective Theory”
arXiv:1908.01493