# The Road from 2 to 7 Loops in Planar N=4 SYM

**Lance Dixon (SLAC)**

Breakthroughs in QFT and Gravity
Queen Mary University of London
7 November 2019

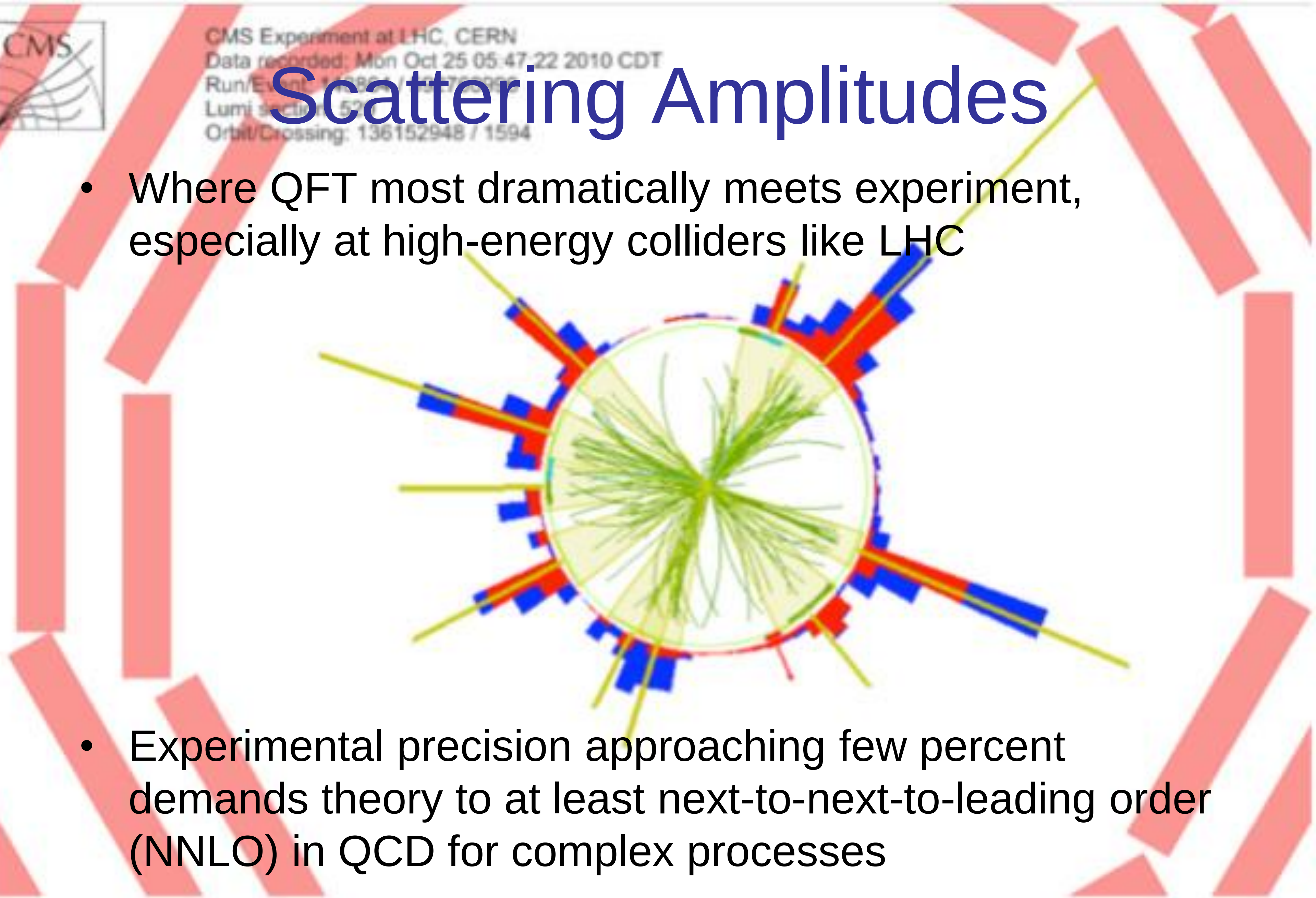
SAGEX
Scattering Amplitudes: from Geometry to Experiment

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 764850 (SAGEX)

SLAC
NATIONAL ACCELERATOR LABORATORY

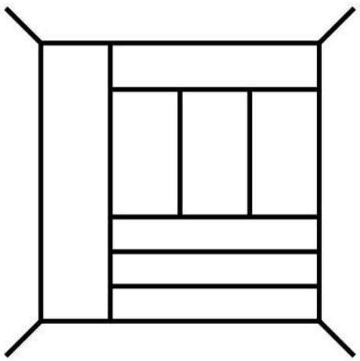
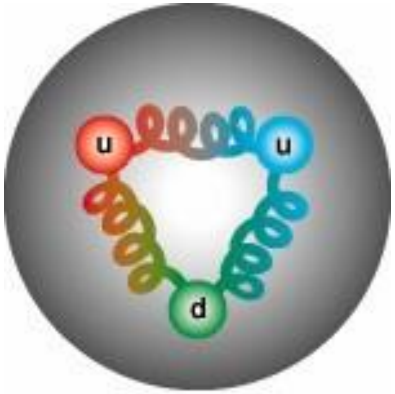
# Road from 2 to 7 passes through many other loops…

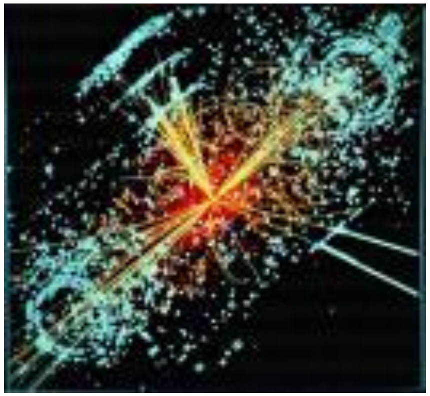
# Scattering Amplitudes




- Where QFT most dramatically meets experiment, especially at high-energy colliders like LHC
- Experimental precision approaching few percent demands theory to at least next-to-next-to-leading order (NNLO) in QCD for complex processes

# N=4 Super-Yang-Mills Theory
# QCD and Collider Physics

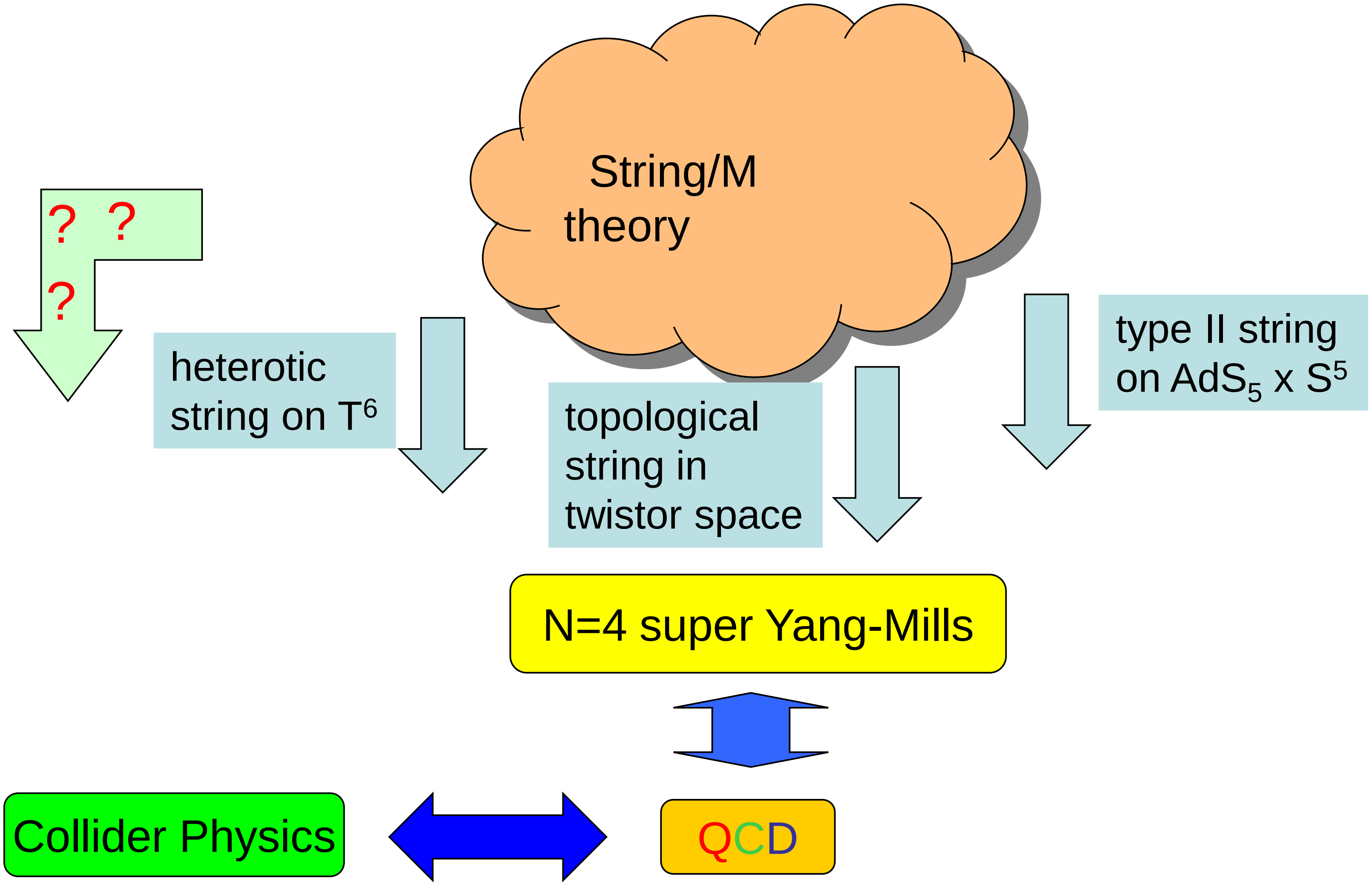





Lance Dixon, SLAC

Queen Mary, University of London
27 September 2004

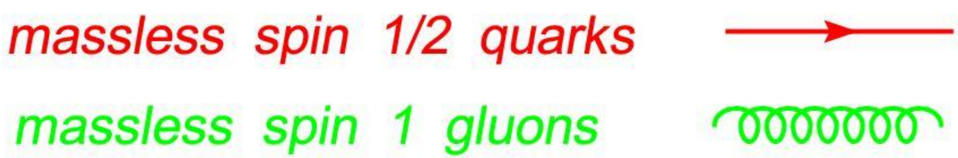
# String theory intertwined too



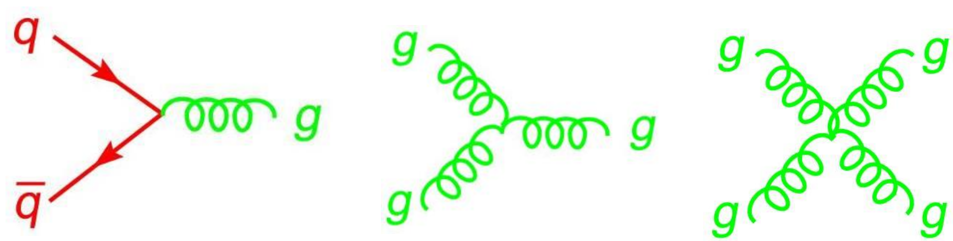
# What is QCD?

- Theory of the strong force, which
  - binds quarks and gluons into nucleons
  - dominates both “hard” and “soft” interactions of nucleons, e.g. pp @ 14 TeV (LHC)

The players:




Their interactions:

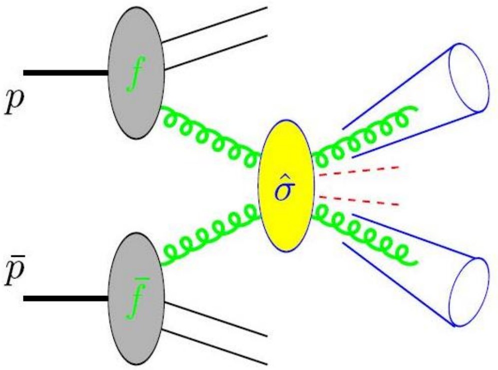



*dimensionless* coupling constant, $\alpha_s = g_s^2/4\pi$

# Why QCD and Collider Physics?

- Particle physics energy frontier at hadron colliders
  - Tevatron now, LHC in 2007
- New physics contends with Standard Model backgrounds
- All physics processes at hadron colliders – signals & backgrounds – are QCD processes
- Basic picture (QCD factorization): uncalculable parts of short-distance processes belong to hadron "wave function" – parton distribution functions *f(x)* *(f = q, g)*

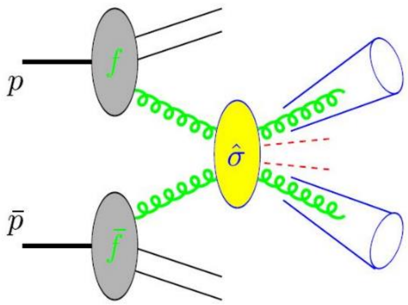
"Partonic cross section" $\hat{\sigma}$ can be computed as perturbative series in $\alpha_s$



# Precision at Colliders?

QCD factorization formula:

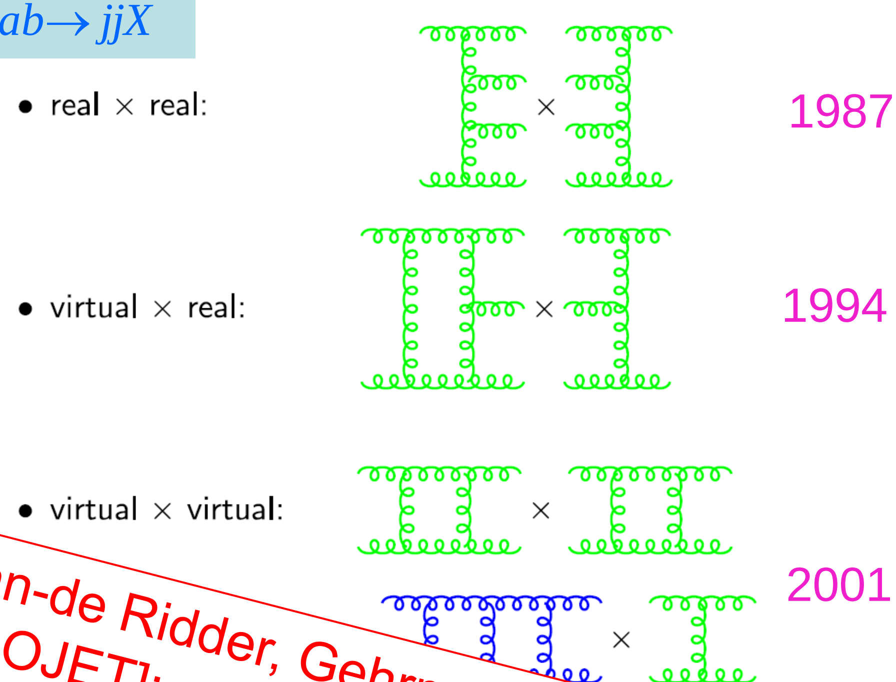
$$\sigma_{p\bar{p}\to jjX} = \sum_{a,b} \int dx_1 dx_2\, f_a(x_1;\mu_F)\, \bar{f}_b(x_2;\mu_F) \times \hat{\sigma}_{ab\to jjX}(sx_1x_2;\mu_F,\mu_R,\alpha_s(\mu_R))$$




- Leading-order (LO) in $\alpha_s$ (tree graphs for $\hat{\sigma}$) only a qualitative estimate
- NLO (1-loop) begins to be quantitative
- Precise predictions only begin at NNLO (2-loops)

# Improving Jet Predictions

## Ingredients for NNLO $\hat{\sigma}_{ab\to jjX}$




- Severe infrared divergences due to massless gluons
- Regulate with $D=4-2\varepsilon$ (br... conformal invariance) ge...les as $\varepsilon \to 0$
- ...
(a...
- Numerical p... due to complicated r... phase-space integrals

Accomplished by Currie, Gehrmann-de Ridder, Gehrmann, Glover, Huss, Pires (…,2017) [NNLOJET]; recently verified by Czakon, van Hameren, Mitov, Poncelet (2019)

- Singular structure of QCD and N=4 super-Yang-Mi... amplitudes *very similar* – one of the ways N=4 SYM serves as a "theoretical playground" for QCD

# N=4 SYM particle content

**The players:**

*massless spin 1 gluon*

*4 massless spin 1/2 gluinos*

*6 massless spin 0 scalars*

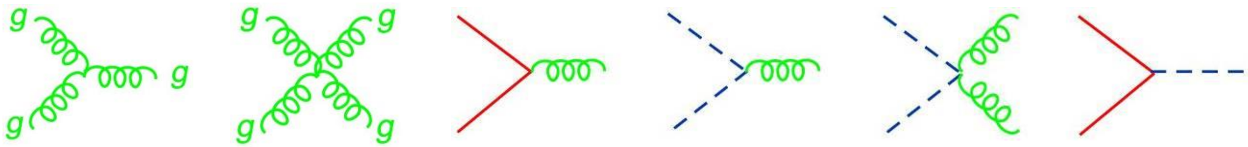
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 1 | | | | |
| | | | 1 | | 1 | | | |
| $\mathcal{N}=2$ | | 1 | | 2 | | 1 | | |
| | 1 | | 3 | | 3 | | 1 | |

| | | | | | |
|---|---|---|---|---|---|
| $\mathcal{N}=4$ | $1 \longleftrightarrow$ | $4 \longleftrightarrow$ | $6 \longleftrightarrow$ | $4 \longleftrightarrow$ | $1$ |
| | $g^-$ | $\lambda^-_{\bar{\imath}}$ | $\bar{\phi}_{\bar{\imath}\bar{\jmath}}, \phi_{ij}$ | $\lambda^+_i$ | $g^+$ |
| helicity | $-1$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ |

all in adjoint representation

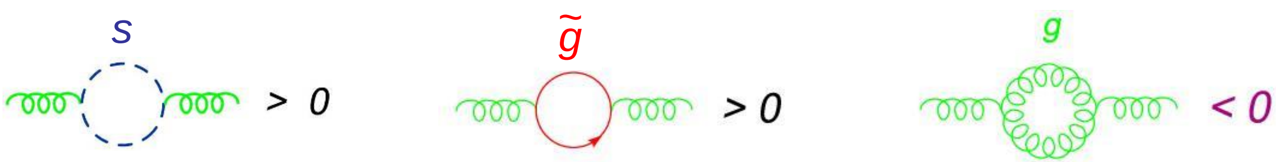
*SUSY*
*$Q_a$, a=1,2,3,4*
*shifts helicity*
*by 1/2* $\longleftrightarrow$

# N=4 SYM interactions




All proportional to same *dimensionless* coupling constant, *g*

- SUSY cancellations: scale invariance *preserved* quantum mechanically



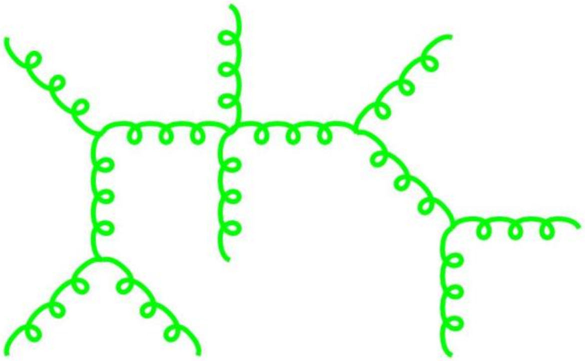

$$\frac{d\alpha}{d\ln\mu^2} = \beta(\alpha) = \left[6\times\frac{1}{6} + 4\times\frac{2}{3} - \frac{11}{3}\right]\frac{N_c\,\alpha^2}{4\pi} = 0 \qquad \text{(true to all orders in } \alpha\text{)}$$

Just the beginning of N=4 “miracles”

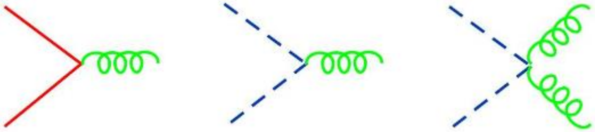
# How are QCD and N=4 SYM related?

## At tree-level they are essentially identical

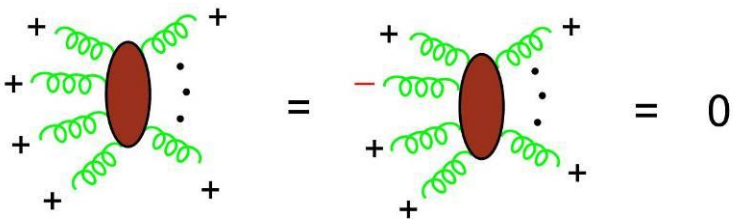
Consider a tree amplitude for *n* gluons

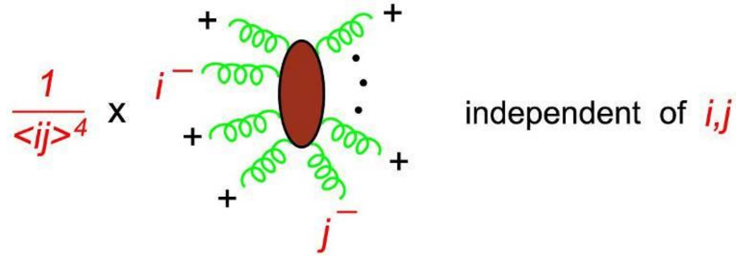


Fermions and scalars cannot appear because they are produced in pairs



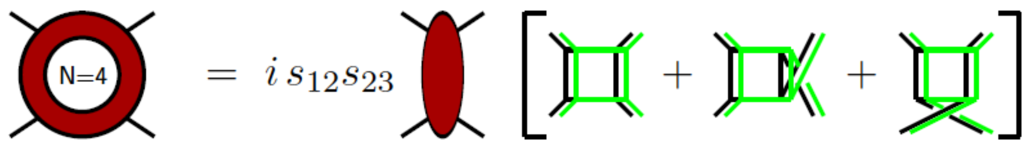
Hence the amplitude is the same in QCD and N=4 SYM. The QCD tree amplitude “secretly” obeys all identities of N=4 supersymmetry:

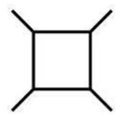


# Simplicity of N=4 SYM 4-point amplitudes

- 1-loop:



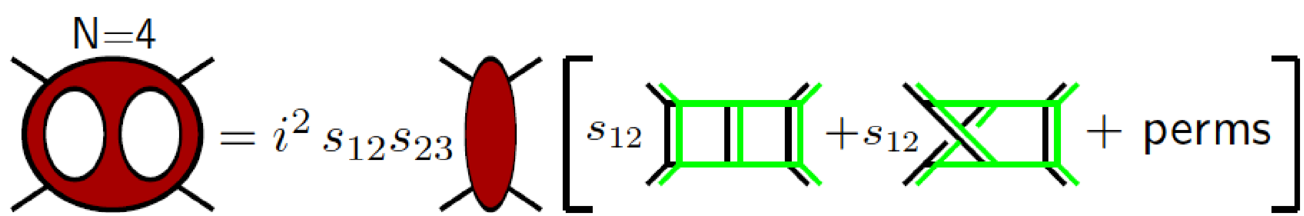

where  $= \int \frac{d^{4-2\epsilon}\ell_1}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell_1^2(\ell_1-k_1)^2(\ell_1-k_1-k_2)^2(\ell_1+k_4)^2}$

 $= \delta^{ab}$  $= f^{abc}$

"color dresses kinematics"

Talk by Zvi

- 2-loops:




Bern, Rozowsky, Yan (1997)

- Analogous computation in QCD not completed until 2001

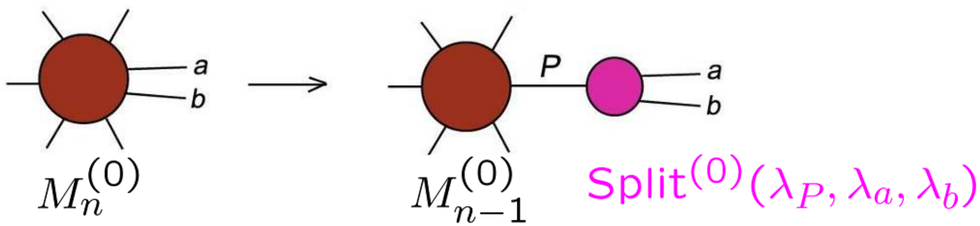
Glover, Oleari, Tejeda-Yeomans (2001); Bern, De Freitas, LD (2002)

# Two-loop exponentiation & collinear limits

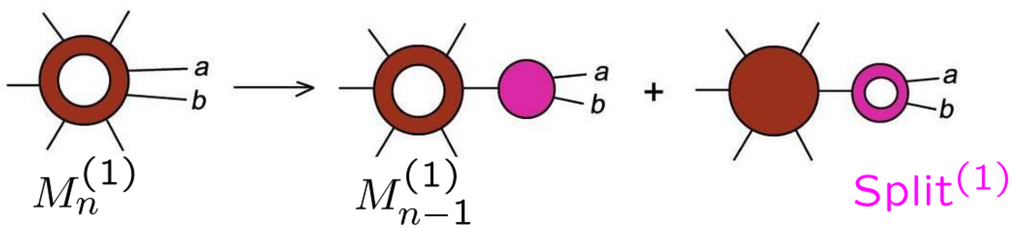
- Evidence for *n>4*: Use limits as 2 momenta become **collinear:**

$$k_a \to z k_P$$
$$k_b \to (1-z) k_P$$

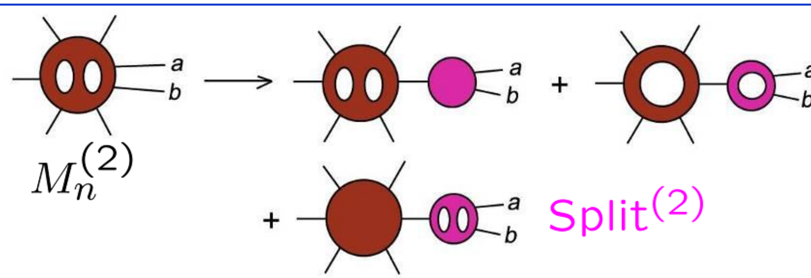
- Tree amplitude behavior:




- One-loop behavior:




- Two-loop behavior:



# Two-loop splitting amplitude iteration and an amplitude conjecture

- In N=4 SYM, all helicity configurations are equivalent, can write

$$\text{Split}^{(l)}(\lambda_P,\lambda_a,\lambda_b) = r_S^{(l)}(z, s_{ab}, \epsilon) \times \text{Split}^{(0)}(\lambda_P,\lambda_a,\lambda_b)$$

- Found that two-loop splitting amplitude obeys:

Anastasiou, Bern, LD, Kosower, hep-th/0309040

$$\boxed{r_S^{(2)}(\epsilon) \;=\; \frac{1}{2}\left[r_S^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon) r_S^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)}$$

where $f^{(2)}(\epsilon) = -\zeta_2 - \epsilon\,\zeta_3 - \epsilon^2\,\zeta_4$

consistent with the $n$-point amplitude ABDK ansatz

$$\mathcal{M}_n^{(2)}(\epsilon) \;=\; \frac{1}{2}\left[M_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon) M$$

wrong for $n > 5$ !

constant from 4-point amplitude

# Why is multi-loops so hard?

- Primarily because multi-loop integrals are intricate, transcendental, multi-variate functions
- In contrast, at one loop all integrals are reducible to scalar box integrals + simpler

→ combinations of dilogarithms

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

+ logarithms and rational terms

't Hooft, Veltman (1974)

# Planar N=4 SYM, a toy model

- QCD's maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, in the large $N_c$ (planar) limit
- Structure is very rigid:

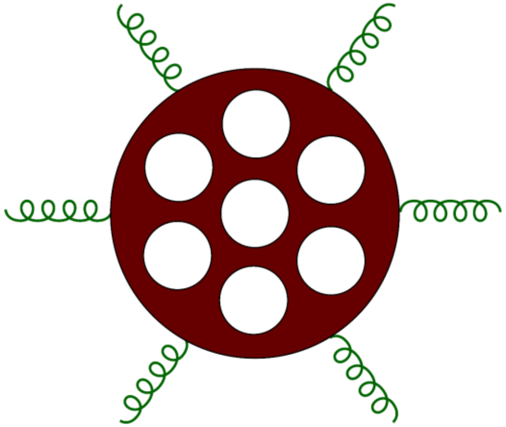
  Amplitudes = $\sum_i rational_i \times transcendental_i$
- For planar N=4 SYM, understand the rational structure quite well, so can focus on the transcendental functions.
- Space of functions is so restrictive, and physical constraints are so powerful, that one can write the $L$ loop answer as a linear combination of weight $2L$ polylogarithms.
- Unknown coefficients found by solving (a large number of) linear constraints,

# Hexagon function bootstrap

3 LD, Drummond, Henn, 1108.4461, 1111.1704;

4,5 Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;

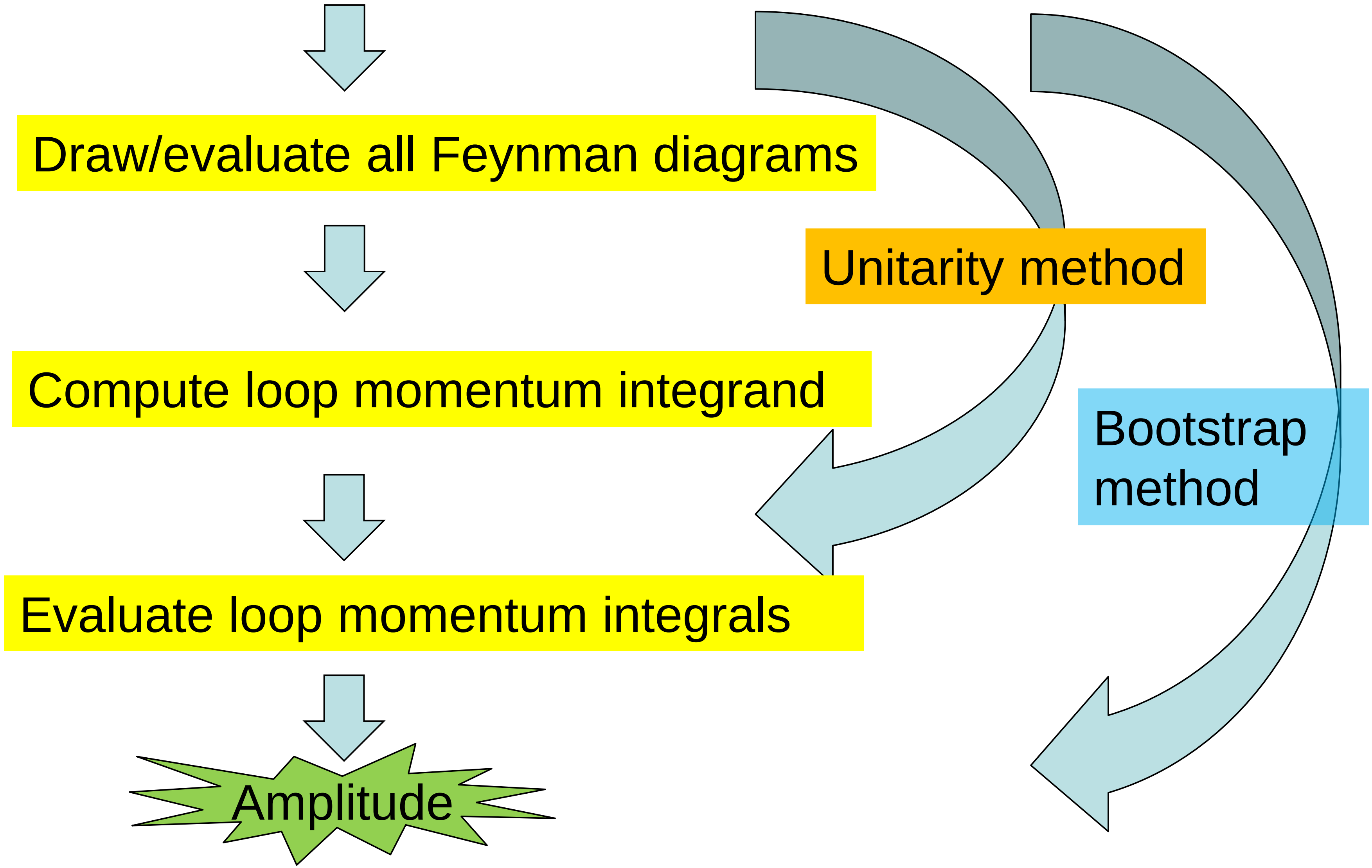
6,7 Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890 and 1906.07116



Use analytical properties of perturbative amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops

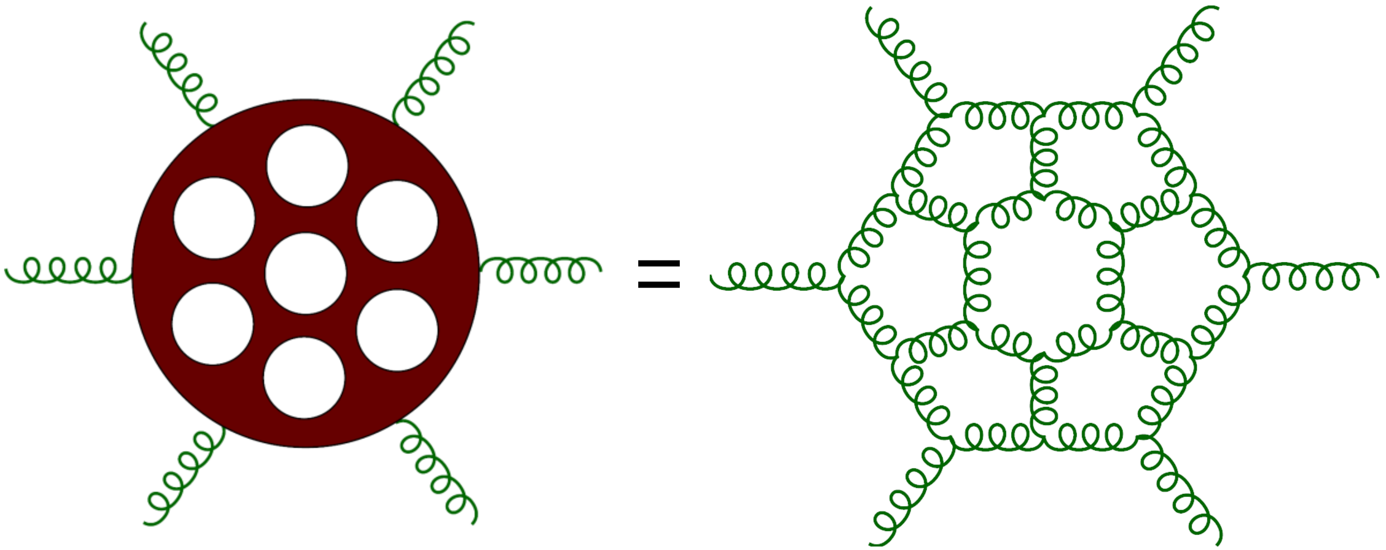
First step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

# Usual approaches vs. bootstrap



# How far can we go?

- So far, through 7 loops



+ ~$10^9$ more

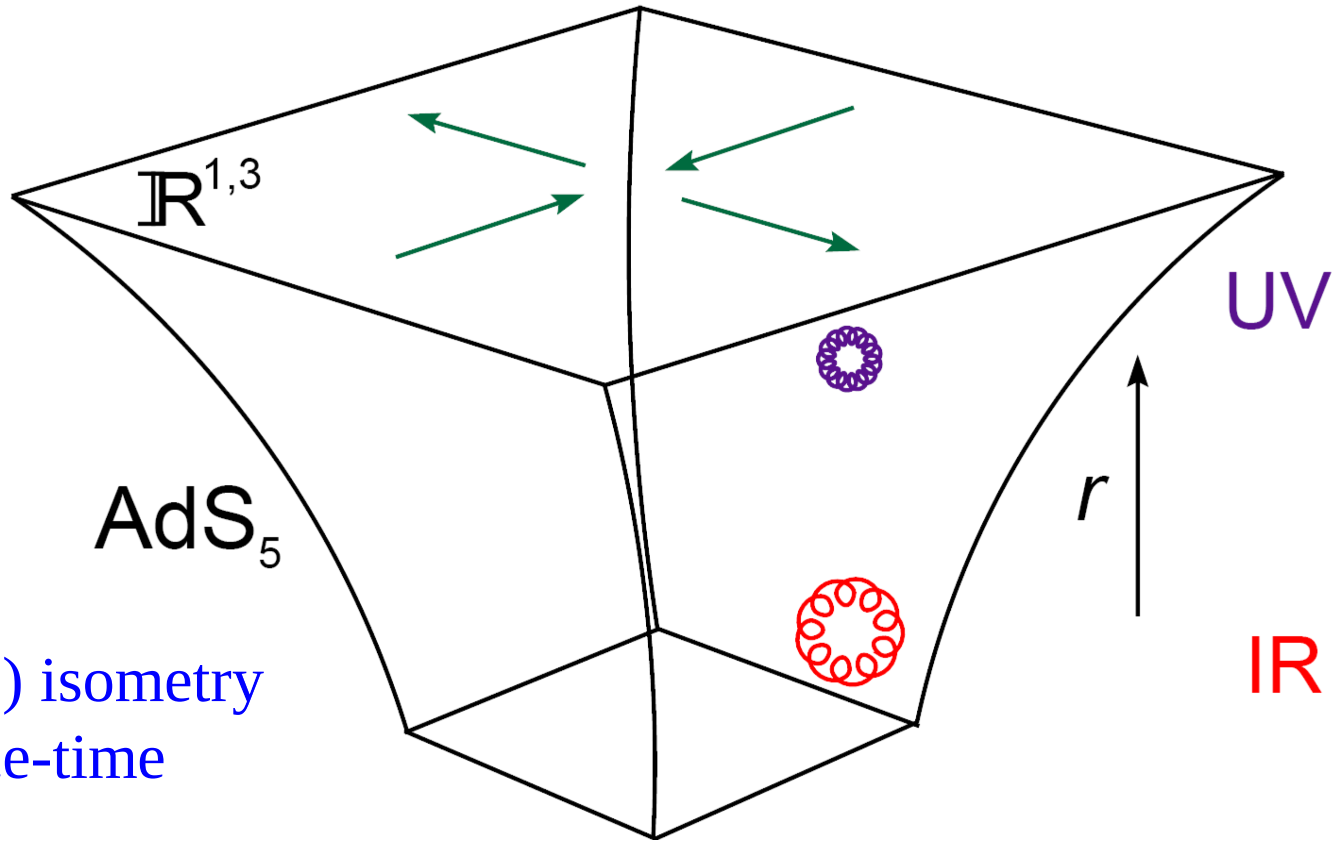
# Quantum Symmetries

- Massless QCD has classical scale + conformal symmetry: $SO(3,1) \rightarrow SO(4,2)$
- Spoiled at quantum level by nonvanishing $\beta$ function (asymptotic freedom).
- N=4 SYM has $\beta = 0$ → full (position space) $SO(4,2)$, actually full N=4 superconformal algebra, $PSU(2,2|4)$


- Planar N=4 SYM also has momentum-space version of $SO(4,2)$ $[PSU(2,2|4)]$
  → dual N=4 superconformal invariance

# Dual conformal invariance is geometric: from AdS/CFT + T-duality

Alday, Maldacena, 0705.0303



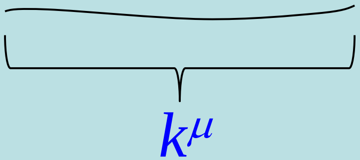

SO(4,2) isometry
of space-time

# T-duality symmetry of string theory

Alday, Maldacena, 0705.0303

- Exchanges string world-sheet variables $\sigma, \tau$
- $X^\mu(\tau,\sigma) = x^\mu + k^\mu \tau$ + oscillators


→ $X^\mu(\tau,\sigma) = x^\mu + k^\mu \sigma$ + oscillators



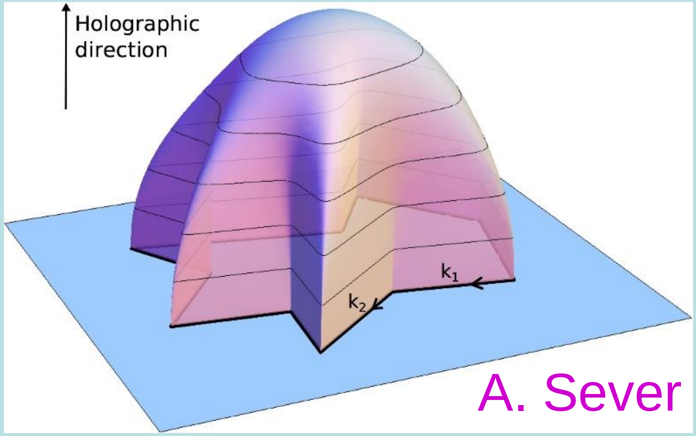

- Strong coupling limit of planar gauge theory

is semi-classical limit of string theory:

world-sheet stretches tight around

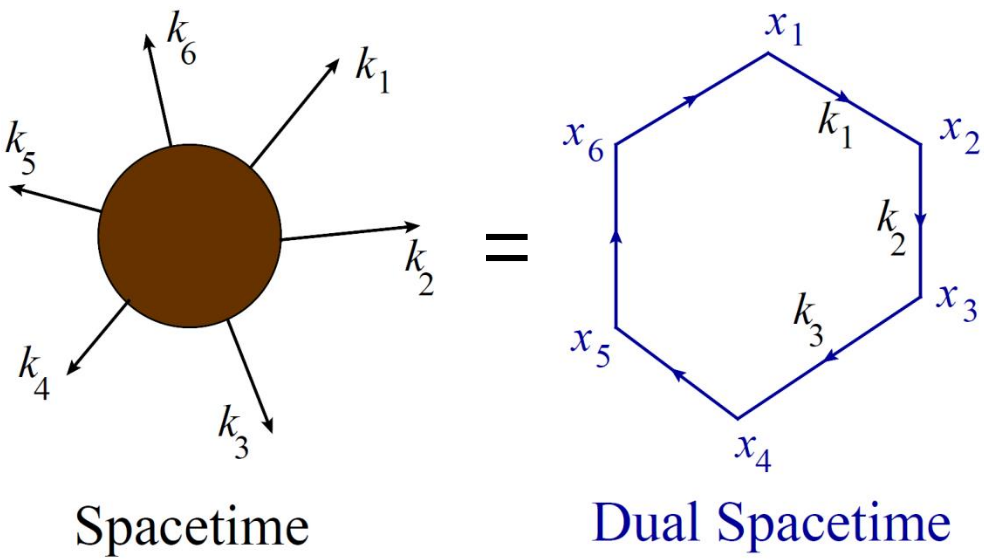
minimal area surface in AdS.

- Boundary determined by momenta of external states: light-like polygon with null edges = momenta $k^\mu$




A. Sever

# Amplitudes = Wilson loops




- Polygon vertices $x_i$ are not positions but dual momenta,
  $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev,
0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin,
Vergu, Volovich, 0803.1465

Duality verified to hold at weak coupling too!

# The [Dual] Conformal Group

SO(4,2) $\supset$ SO(3,1) [rotations+boosts] + translations+dilatations + special-conformal

15 = 3 + 3 + 4 + 1 + 4

- Nontrivial generators are special conformal $K^\mu$
- Correspond to inversion · translation · inversion
- To obtain a [dual] conformally invariant function $f(x_{ij}^2)$ just have to check invariance under inversion,

$$x_i^\mu \to x_i^\mu / x_i^2$$

# Dual conformal invariance

- Wilson *n*-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2\, x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

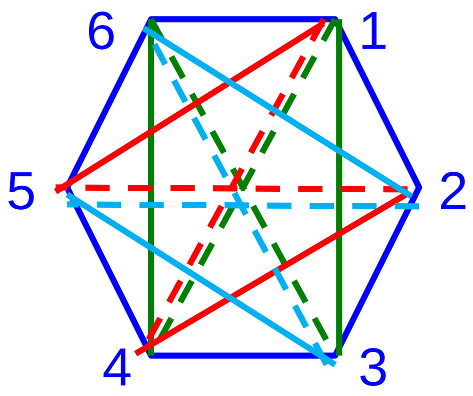
$$\boxed{u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}}$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for *n* = 4,5

*n* = 6 → precisely 3 ratios:

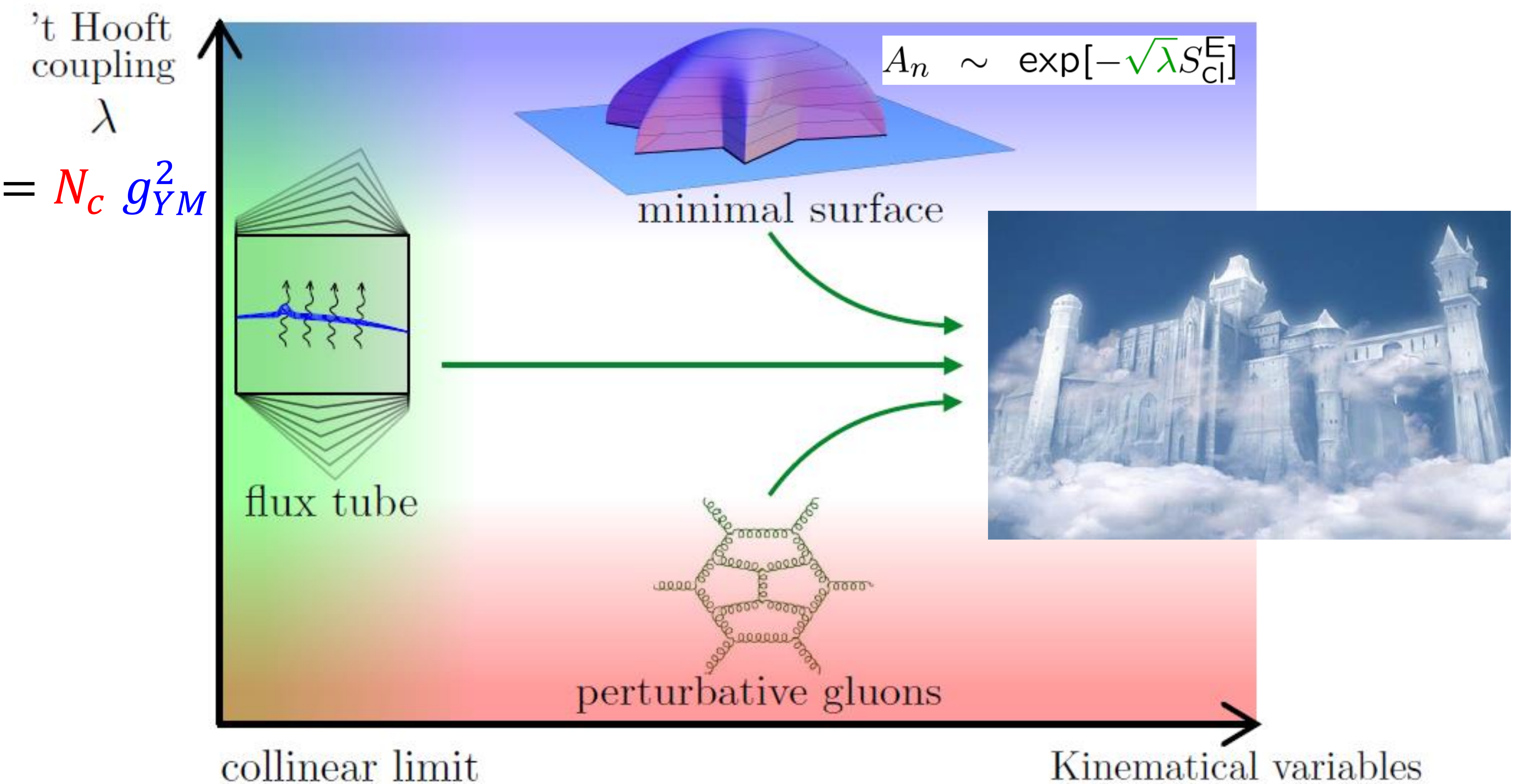
$$\left\{ \begin{aligned} u &= \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}} \\ v &= \frac{s_{23}s_{56}}{s_{234}s_{123}} \\ w &= \frac{s_{34}s_{61}}{s_{345}s_{234}} \end{aligned} \right.$$

In general, 3*n*-15 ratios



# Solving Planar N=4 SYM Scattering

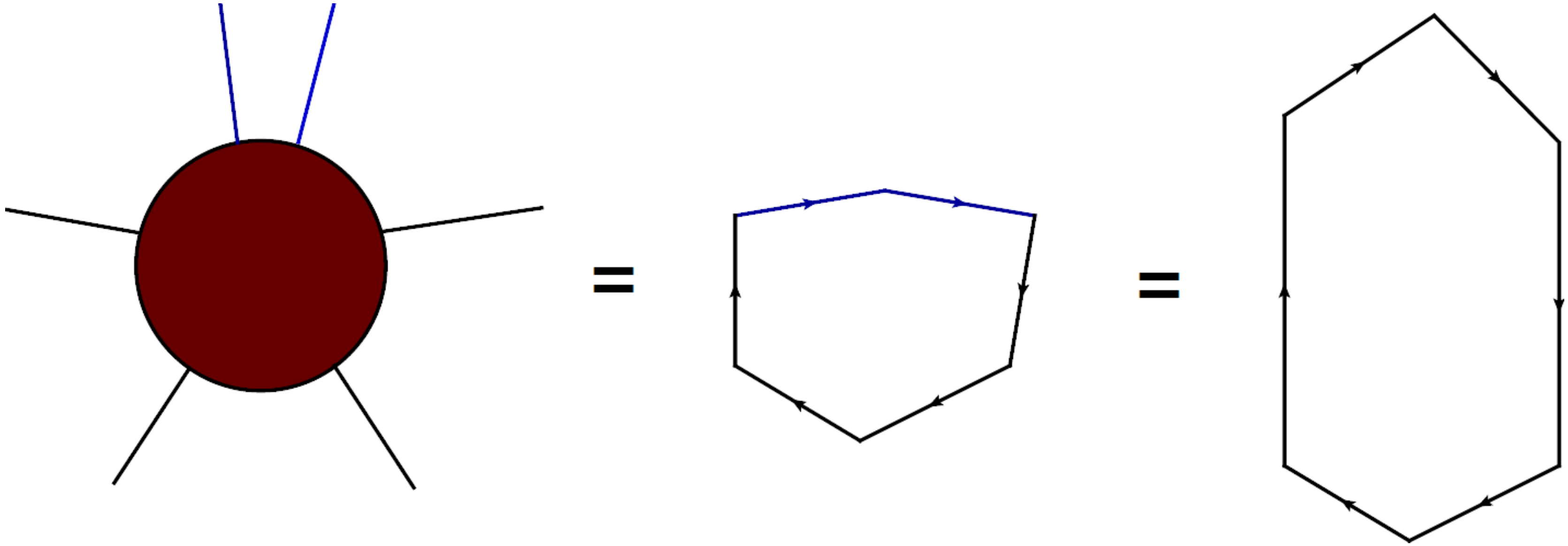
Images: A. Sever, N. Arkani-Hamed



# Rich theoretical "data" mine

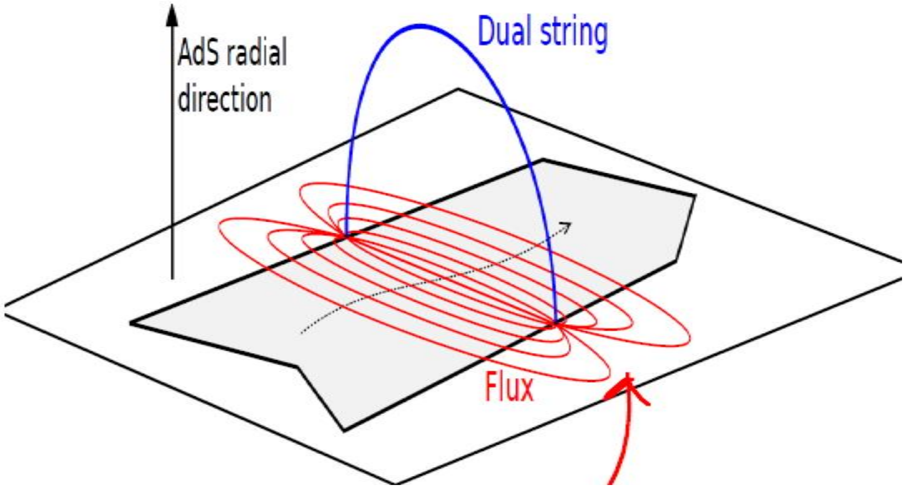

- Rare to have perturbative results to 6 or 7 loops
- Usually high loop order → single numbers such as $\beta$ functions or anomalous dimensions
- Here we have analytic functions of 3 variables (6 variables in 7-point case)
- Many limits to study (and exploit)

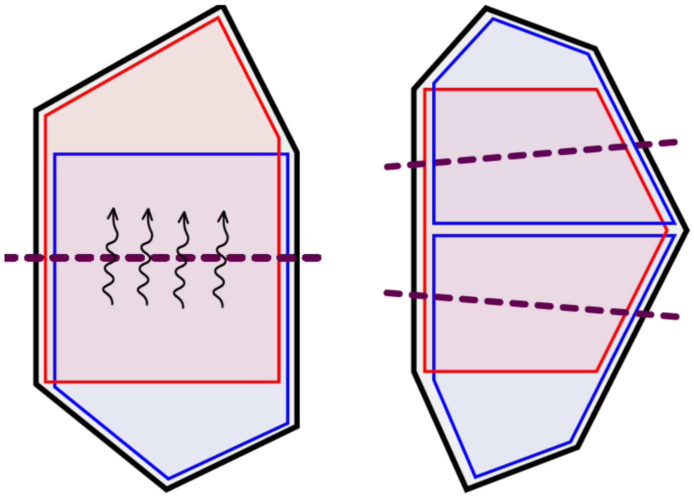
# (Near) collinear (OPE) limit



# Flux tubes at finite coupling

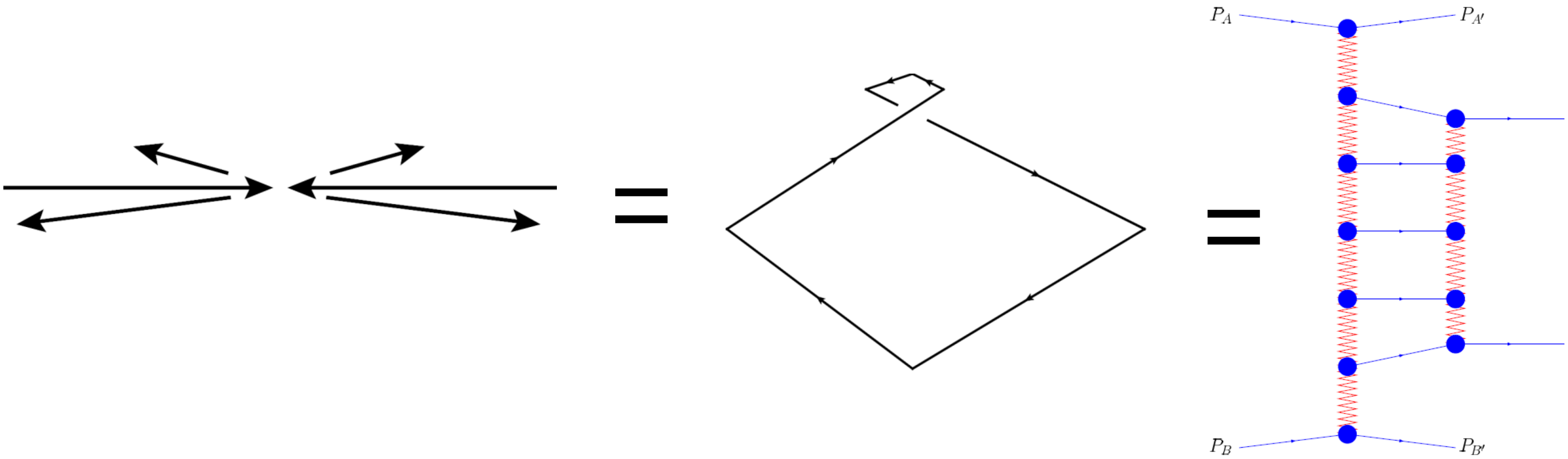
Alday, Gaiotto, Maldacena,  Sever, Vieira,  1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987



- Tile $n$-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in ’t Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations =  expansion around near collinear limit

# Multi-regge limit

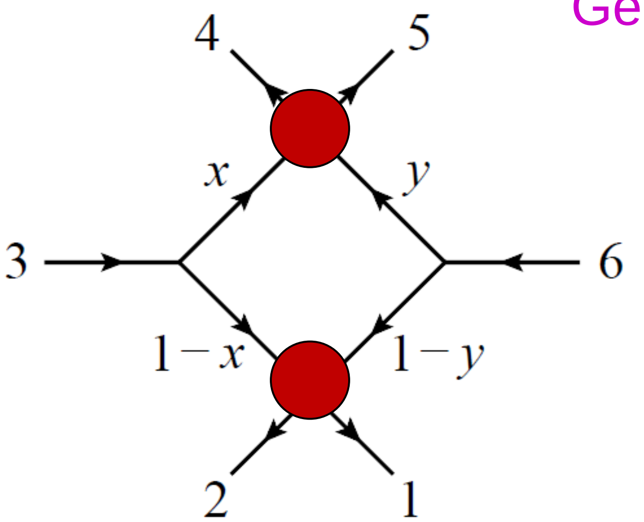



- Amplitude factorizes in Fourier-Mellin space

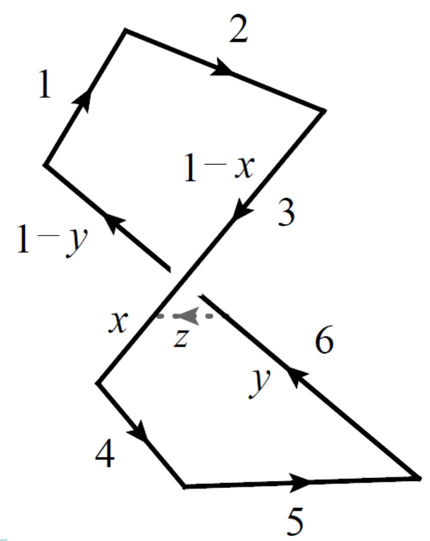
Bartels, Lipatov, Sabio Vera, 0802.2065, Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186; Pennington, 1209.5357;
Basso, Caron-Huot, Sever, 1407.3766 (analytic continuation from OPE limit);
Broedel, Sprenger, 1512.04963; Lipatov, Prygarin, Schnitzer, 1205.0186;
LD, von Hippel, 1408.1505; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek, 1606.08807;…

# Double-parton-scattering-like limit

Georgiou, 0904.4675; LD, Esterlis, 1602.02107



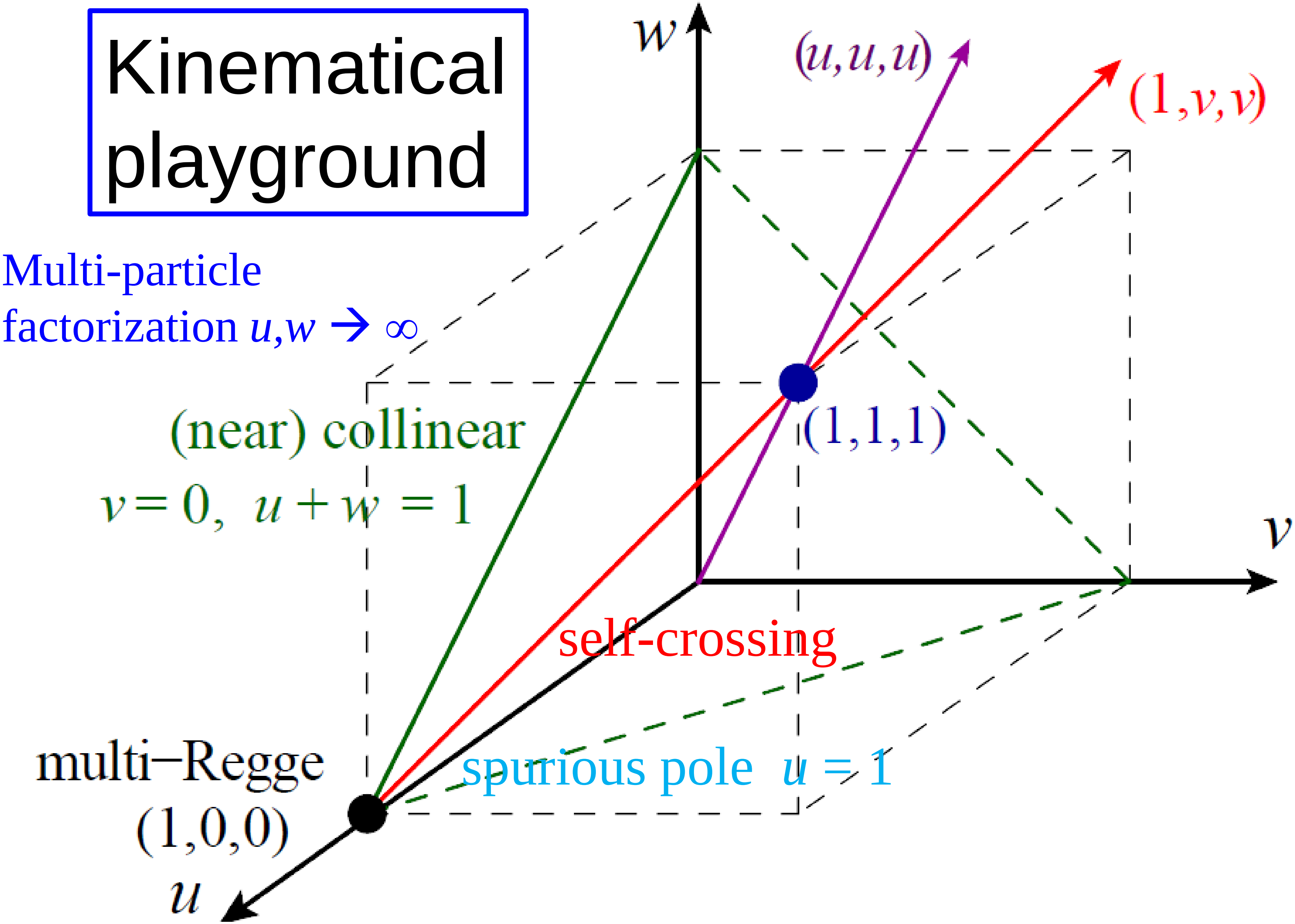

=




- Self-crossing limit of Wilson loop, $\delta \sim |z|^2 \rightarrow 0$
- Overlaps MRK limit
- A virtual Sudakov region, $A \sim \exp[-\ln^2\delta]$
- Singularities ~ Wilson line RGE

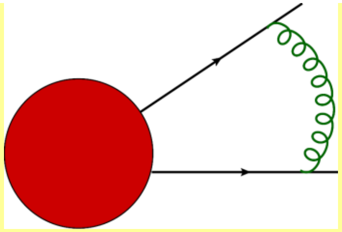
Korchemsky and Korchemskaya hep-ph/9409446

# Kinematical playground

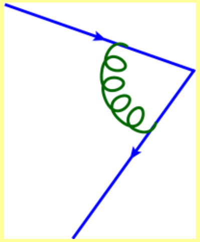


# IR Renormalization Schemes

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\gamma_K$
  **known to all orders in planar N=4 SYM:**
  Beisert, Eden, Staudacher, hep-th/0610251



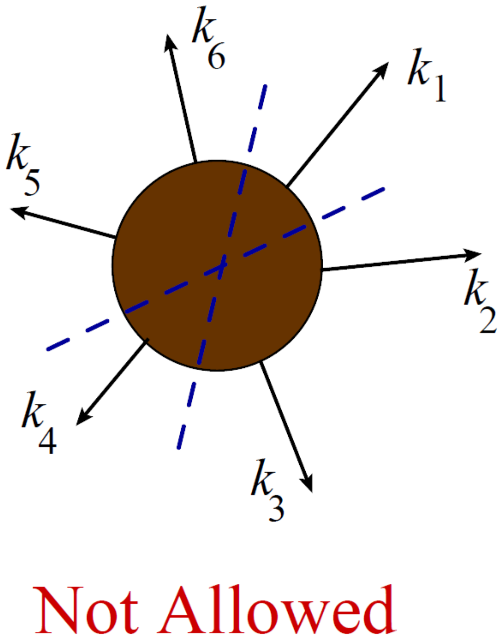
- Both removed by dividing by either BDS ansatz or BDS-like ansatz Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708

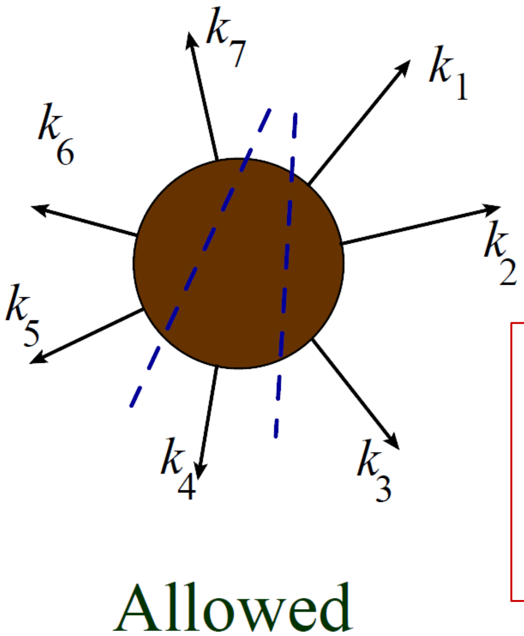
- Normalized amplitude finite, dual conformal invariant.
- BDS-like → also maintains important relation due to causality (Steinmann)

# Steinmann relations

Steinmann, Helv. Phys. Acta (1960)   Bartels, Lipatov, Sabio Vera, 0802.2065

- Amplitudes should not have overlapping branch cuts:




can't apply to 2 particle cuts in massless case because they are not independent

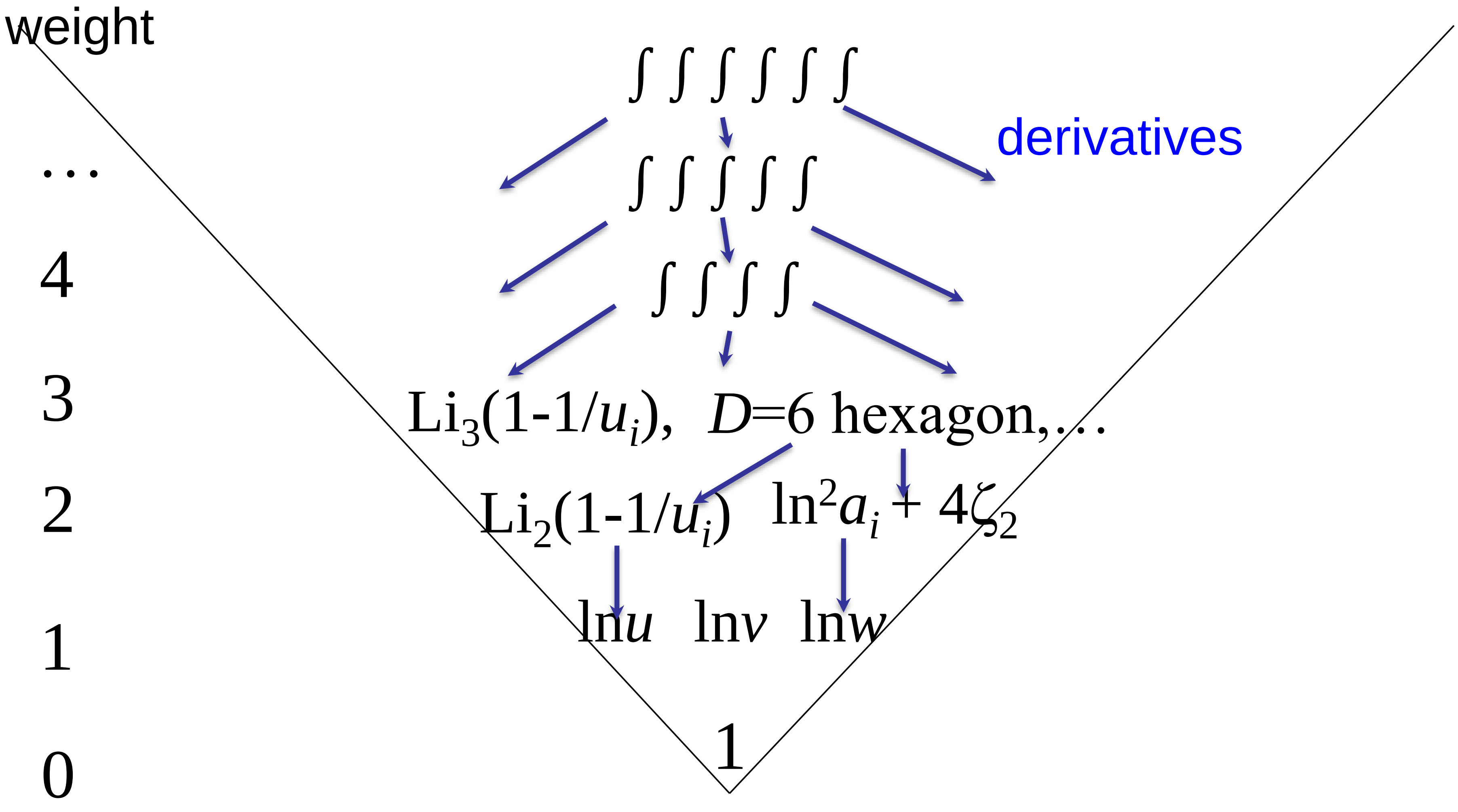
$$\boxed{\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0}$$

Violated by ABDK and BDS ansatz!

# Key “initial” condition

- Two-loop 6-gluon result first computed numerically from both amplitude and Wilson loop pictures Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1466; Drummond, Henn, Korchemsky, Sokatchev, 0803.1466
- Wilson loop side then evaluated analytically → 17 pages of [Goncharov] polylogarithms Del Duca, Duhr, Smirnov, 0911.5332, 1003.1702
- Simplified to a few lines in term of classical polylogs $\mathrm{Li}_n(x)$, demonstrating power of symbol Goncharov, Spradlin, Vergu, Volovich, 1006.5703
- Told us what types of functions were likely to occur at higher loops. (For experts: symbol alphabet.)

# Heuristic view of space



# Master Table

(MHV,NMHV): parameters left in $(\mathcal{E}^{(L)},\ E^{(L)} \& \tilde{E}^{(L)})$

| Constraint | $L=1$ | $L=2$ | $L=3$ | $L=4$ | $L=5$ | $L=6$ |
|---|---|---|---|---|---|---|
| 1. All functions | (6,6) | (25,27) | (92,105) | (313,372) | (991,1214) | (2951,3692?) |
| 2. Symmetry | (2,4) | (7,16) | (22,56) | (66,190) | (197,602) | (567,1795?) |
| 3. Final entry | (1,1) | (4,3) | (11,6) | (30,16) | (85,39) | (236,102) |
| 4. Collinear limit | (0,0) | (0,0) | $(0^*, 0^*)$ | $(0^*, 2^*)$ | $(1^{*3}, 5^{*3})$ | $(6^{*2}, 17^{*2})$ |
| 5. LL MRK | (0,0) | (0,0) | (0,0) | (0,0) | $(0^*, 0^*)$ | $(1^{*2}, 2^{*2})$ |
| 6. NLL MRK | (0,0) | (0,0) | (0,0) | (0,0) | $(0^*, 0^*)$ | $(1^*, 0^*)$ |
| 7. NNLL MRK | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (1,0) |
| 8. $N^3$LL MRK | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (1,0) |
| 9. all MRK | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (1,0) |
| 10. $T^1$ OPE | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (1,0) |
| 11. $T^2 F^2 \ln^4 T$ OPE | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) |
| 12. all $T^2 F^2$ OPE | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) |

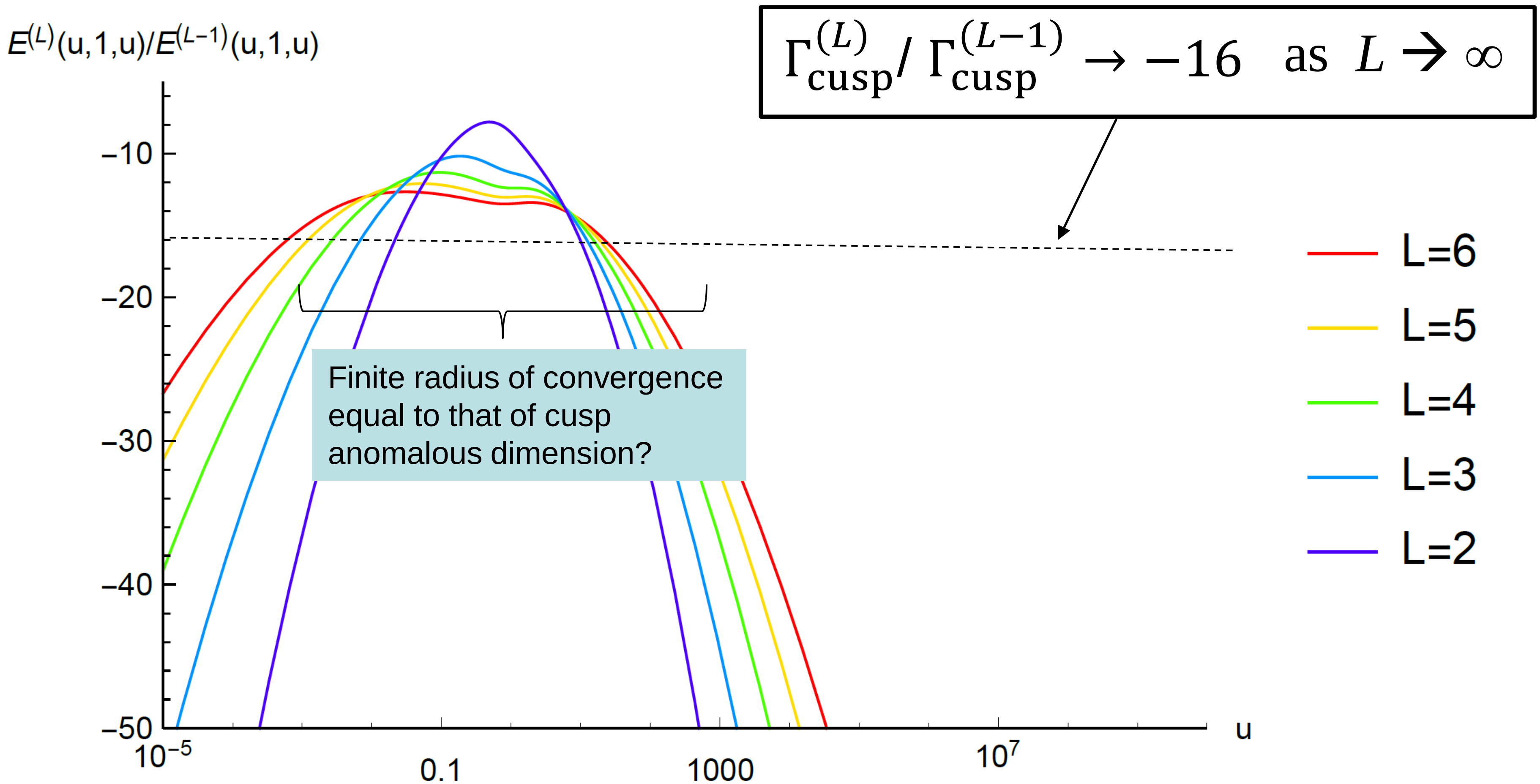
(0,0) → amplitude uniquely determined

Also have MHV at $L = 7$

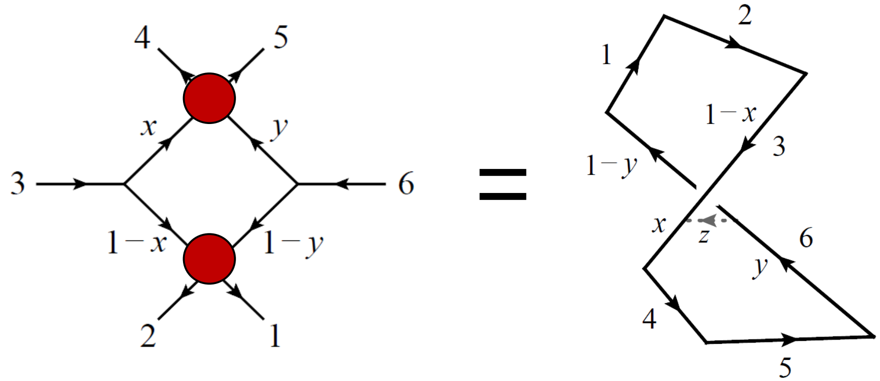
# Properties of Amplitudes

- Having determined the 6-point amplitudes to 6 (7) loops for NMHV (MHV), can study their physical, numerical and (number-theoretic) properties.
- Analytic behavior in various factorization limits.
- What kinds of transcendental numbers appear?
- Numerics feasible on "simple lines" like $(u,u,1)$, $(u,1,1)$, $(u,u,u)$.
- Planar N=4 SYM should have finite radius of convergence of perturbative expansion (unlike QCD, QED, whose perturbative series are asymptotic).
- For BES solution to cusp anomalous dimension, using coupling $g^2 = \frac{\lambda}{16\,\pi^2}$, radius is $\frac{1}{16}$
- Ratio of successive coefficients $\gamma_K^{(L)} / \gamma_K^{(L-1)} \to -16$

# NMHV Amplitude on $(u,u,1)$



# Self-crossing resummation




1903.10890

$$u \to u e^{-2\pi i}$$
$$(u, v, w) \to (1 - |\delta|, v, v)$$

- Exact result for all leading-power terms as $\delta \to 0$:

$$\frac{1}{2\pi i}\frac{d\mathcal{E}_{3\to 3}}{d\ln|\delta|} = \frac{g^2}{\rho}\exp\left[\tfrac{1}{2}\zeta_2\Gamma_{\rm cusp} + 2\Gamma_3\right] \times 2\int_0^\infty d\nu\, J_1(2\nu)\exp\left[-\frac{1}{4}\Gamma_{\rm cusp}[\lambda(\nu)]^2 - \Gamma_{\rm virt}\lambda(\nu)\right]$$

$$\lambda(\nu) = 2(\ln\nu + \gamma_E) - \ln|\delta|$$

$\Gamma_{\rm cusp}$, $\Gamma_{\rm virt}$, $\Gamma_3$ known exactly in ‘t Hooft coupling

Beisert, Eden, Staudacher hep-th/0610251; Basso, 1010.5237

- checked explicitly through 7 loops

# At $(u,v,w) = (1,1,1)$, amplitude → MZVs

Allowed MZV's obey a Galois "co-action" principle, restricting the combinations that can appear
Brown, Panzer, Schnetz

MHV

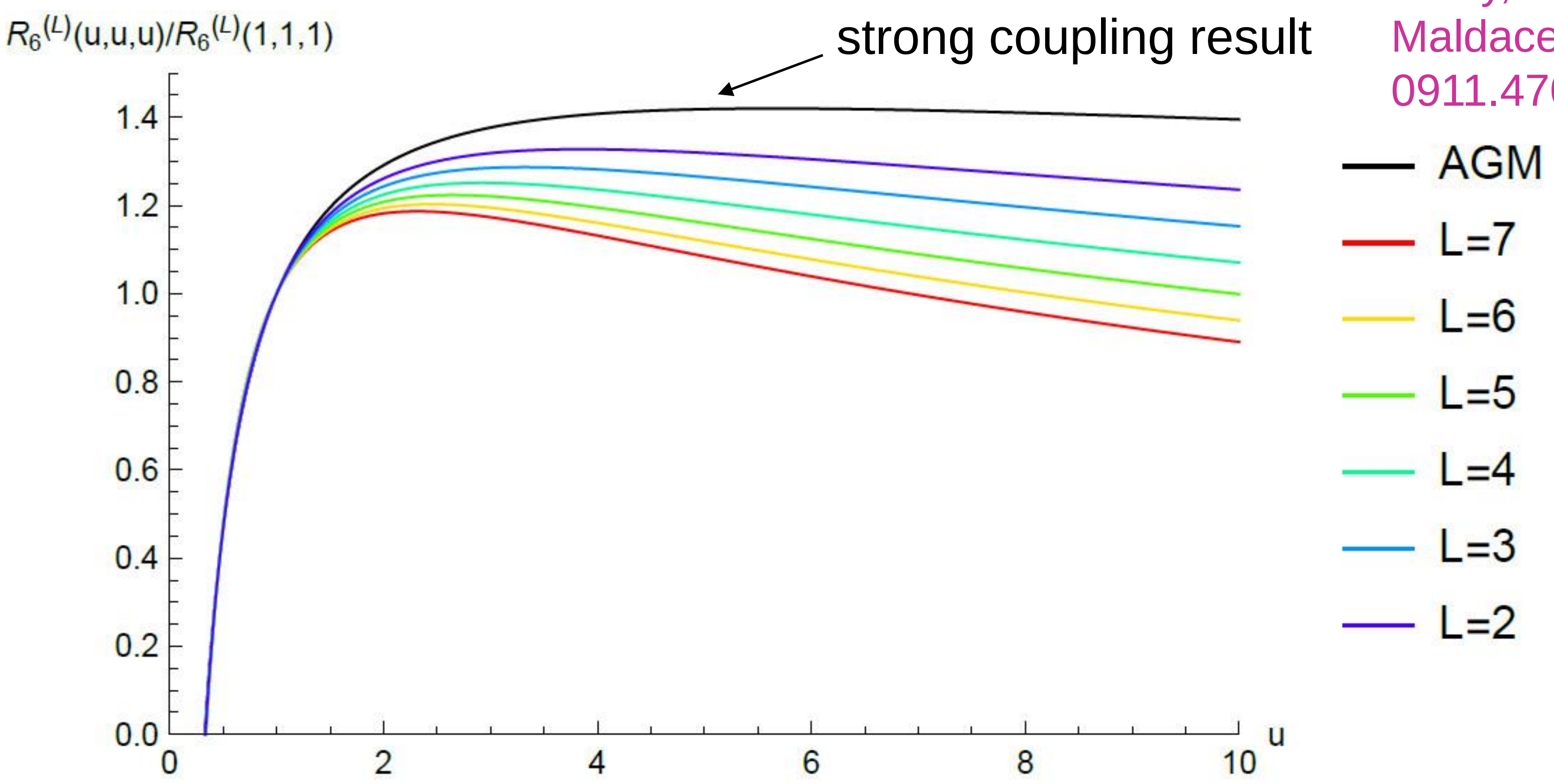
$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] - 96\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

NMHV

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = -\frac{36271}{9}\,\zeta_8 - 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] + 132\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

# MHV numerical values on $(u,u,u)$

Alday, Gaiotto, Maldacena, 0911.4708




- Why such an **amazing proportionality** of **each** perturbative coefficient at small $u$, and also with the strong coupling result???

# High loop orders beyond 6 gluons

- Cluster algebras provide strong clues to right polylogarithmic function space
  Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Foster, Gurdogan, 1710.10953
- Symbol of 3-loop MHV 7-point amplitude bootstrapped first. 42 letter alphabet.
  More rigid: No need for OPE constraints
  Drummond, Papathanasiou, Spradlin 1412.3763
- With Steinmann relations, could go to 4-loop MHV and 3-loop non-MHV LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976
- Extended Steinmann → 4-loop NMHV
  Drummond, Gurdogan, Papathanasiou, 1812.04640
- Still need to go from symbols → actual functions!

# Summary & Outlook

- Planar N=4 SYM scattering amplitudes/Wilson Loops determined to high loop order, simply by writing linear combination of right functions and imposing a few boundary constraints.
- Rich information about many different kinematic limits.
- Large order behavior reflects finite radius of convergence (~ that of $\Gamma_{\text{cusp}}$) for $u,v,w \sim 1$.
- Interesting number-theoretic restrictions.
- Next challenge: go to finite coupling for generic kinematics! What are the right finite-coupling functions? Clues from OPE/integrability?
- How many lessons can we apply to QCD?

# The road goes on!